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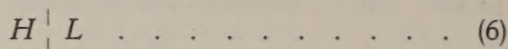
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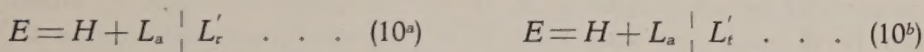
CONTENTS

- F. A. H. SCHREINEMAKERS: "Equilibria in systems, in which phases, separated by a semi-permeable membrane". XII, p. 858.
- J. D. VAN DER WAALS Jr.: "The Equilibrium in the Capillary Layer", p. 868.
- E. VAN EVERDINGEN: "The cyclone-like whirlwinds of August 10, 1925", p. 871.
- ERNST LAQUEUR, P. C. HART, S. E. DE JONGH and I. A. WIJSENBEEK: "On the preparation of the hormone of the estrous cycle, and its chemical and pharmacological properties". (Communicated by Prof. R. MAGNUS), p. 890.
- J. DROSTE: "The analogue of Clapeyron's law in the case of evaporating electrons". (Communicated by Prof. H. A. LORENTZ), p. 900.
- P. H. GALLÉ: "On the relation between wind or current and mean sealevel in the Indian and the Atlantic oceans and the adjacent seas". (Communicated by Dr. J. P. VAN DER STOK), p. 905.
- M. J. VAN UVEN: "On treating Skew Correlation". (Continuation). Communicated by Prof. W. KAPTEIJN), p. 919.
- ERNST COHEN and SABURO MIYAKE: "The influence of minute traces of Water on Solution Equilibria". II, p. 936.
- G. GRIJNS: "Diet and reproduction", p. 942.
- G. H. DIEKE and A. C. S. VAN HEEL: "On the difference of the fluorescence and the absorption spectra of the uranyl salts". (Communicated by Prof. H. KAMERLINGH ONNES), p. 953.
- A. C. S. VAN HEEL: "On the monochromatic excitation of fluorescence. (Communicated by Prof. H. KAMERLINGH ONNES), p. 955.
- A. TH. VAN URK, W. H. KEESOM and H. KAMERLINGH ONNES: "Measurements of the surface tension of liquid helium", p. 958.
- G. P. NIJHOFF and W. H. KEESOM: "Isotherms of di-atomic substances and their binary mixtures. XXXIII. Isotherms of oxygen between -40° C. and $-152^{\circ}.5$ C. and pressures from 3 to 9 atmospheres", p. 963.
- G. J. SIZOO and H. KAMERLINGH ONNES: "Further experiments with liquid helium. CA. On the properties of supraconductive metals in the form of thin films", p. 971.
- B. I. LAWRENTJEW: "Ueber die nervöse Natur und das Vorkommen der sogenannten interstiellen Zellen (Cajal, Dogiel) in der glatten Muskulatur", p. 977.
- PAUL URYSOHN†: "Sur les points accessibles des ensembles fermés". (Communicated by Prof. L. E. J. BROUWER), p. 984.
- L. TUMARKIN: "Zur allgemeinen Dimensionstheorie". (Communicated by Prof. L. E. J. BROUWER), p. 994.
- PAUL ALEXANDROFF: "Ueber stetige Abbildungen kompakter Räume". (Communicated by Prof. L. E. J. BROUWER), p. 997.
- L. TUMARKIN: "Ueber Zerlegungen kompakter metrischer Räume in zueinander fremde abgeschlossene Mengen". (Communicated by Prof. L. E. J. BROUWER), p. 1000.

Let us consider now the osmotic system:

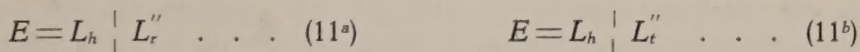


then a equilibrium (7^b) is formed. L_r and L_t , however, have a greater O.Y.A. than the liquids of the isotonic curve ak and, therefore, a greater O.Y.A. than the liquid L_a in (7^b). Consequently L_r and L_t will remove Y from L_a , but L_a will keep its composition by solution of this hydrate, as long as a sufficient quantity of H is present. If a sufficient quantity of solid H is present, then the diffusion continues till the liquids L_r and L_t have reached the isotonic curve ak . The systems (9^a) and (9^b) then pass into the osmotic equilibria:



Herein L'_r and L'_t are the liquids which are represented by the points of intersection of the lines rY and tY with the isotonic curve ak .

If in the systems (9^a) and (9^b) no sufficient quantity of solid H is present, so that it has disappeared already before L_r and L_t have reached the isotonic curve ak , then L_a passes into an unsaturated solution; consequently in fig. 1 it is displaced starting from point a in the direction towards W . The diffusion of the substance Y stops when the liquids come on the same isotonic curve; if this is the case f. i. on curve he , then (9^a) and (9^b) pass into:



in which L''_r and L''_t are liquids which are represented by the points of intersection of the lines rY and tY with the isotonic curve he .

We can summarise the previous results in the following way:

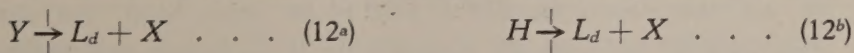
If we bring in osmotic contact the solid hydrate H (fig. 1):

a. with a liquid of the isotonic curve ak then there is osmotic equilibrium;

b. with a liquid of the field $akcb$ then nothing happens;

c. with a liquid of the field $akXW$ then the hydrate will flow away totally or partly.

If we assume that also the component X occurs as solid substance, then we can imagine the saturation-curve of X to be represented in fig. 1 by curve de . As follows from our previous communications, the O.Y.A. of the liquids of this curve increases in the direction of the arrows, viz. from e towards d . We now consider the osmotic systems:



in which, therefore, L_d is a binary liquid, which contains the components water + X and is saturated with solid X . We imagine the complex $L_d + X$ to be represented by the point of intersection m_1 of the line Ym and the side WX , which is not drawn. As the O.Y.A. of L_d is greater than that of solid Y and of solid H , the substance Y will diffuse in (12^a) and (12^b) in the direction of the arrow.

The complex $L_d + X$ moves, therefore, by taking in Y , starting from m_1 towards point Y . Consequently liquid L_d firstly traces curve dm and afterwards the line mY . If a sufficient quantity of solid Y and H is present, then (12^a) and (12^b) pass into the osmotic equilibria:

$$E = Y \mid L_z \quad \dots \quad (13^a)$$

$$E = H + L_a \mid L_v \quad \dots \quad (13^b)$$

One of the visible results of the diffusion Y is, therefore, the disappearance of the solid substance X . In (13^a) is formed the liquid L_z , which can be in equilibrium with solid Y , in (13^b) the unsaturated solution L_v , which is situated on the isotonic curve ak .

The conversion of (12^a) into (13^a) would take place also, if we take away the membrane; if, however, we take away the membrane in (12^b), then it is clear that (13^b) can not be formed.

In the osmotic system:

$$\text{water} \leftarrow L_v \quad \dots \quad (14)$$

the substance Y will diffuse from each liquid L_v towards the water. Then the water forms a binary liquid, which moves along the side WY in the direction towards Y . This diffusion continues till both liquids reach the same isotonic curve; if this is f.i. curve no , then (14) passes into the osmotic equilibrium:

$$E = L_n \mid X + L_0 \quad \dots \quad (15)$$

The visible result of the diffusion of the substance Y is, therefore, the separation of solid X from a liquid, originally unsaturated; this separation begins, as soon as the liquid L_v has reached the point m of the saturation-curve de .

We now consider the osmotic system:

$$X \mid L \quad \dots \quad (16)$$

in which L is an arbitrary liquid. If a little Y diffuses from the liquid towards the left, then is formed there a little of the liquid L_e (fig. 1); then system (16) passes into:

$$X + L_e \mid L \quad \dots \quad (17)$$

The O.Y.A. of the solid substance X is equal, therefore, to that of the liquid L_e and consequently equal also to the O.Y.A. of the liquids of the isotonic curve eh . Therefore, all liquids of the region $ehWd$ have a greater O.Y.A. and all liquids of the region $ehabc$ have a smaller O.Y.A. than the solid substance X .

Hence follows:

If we bring in osmotic contact the solid substance X (fig. 1):

- a. with a liquid of the isotonic curve eh then there is osmotic equilibrium;
 b. with a liquid of the field $ehWd$, then nothing happens; the osmotic system remains unchanged, therefore;
 c. with a liquid of the field $ehabc$ then the solid substance X will flow away totally or partly.

The osmotic systems:

$$X \mid water \quad X \mid L \quad X \mid L_r \quad X \mid L_u \quad X \mid L_t \quad X \mid X + L_m \quad (18)$$

etc., the liquids of which are situated within the field $ehWd$, remain unchanged, therefore, without diffusion occurring. In the osmotic systems:

$$\left. \begin{array}{ccc} X \leftarrow L_a & X \leftarrow H + L_f & X \leftarrow H + Y + L_b \\ X \leftarrow Y + L_q & X \leftarrow L_p & X \leftarrow L_g \end{array} \right\} \quad (19)$$

etc., the liquids of which are situated within the field $ehabc$, the substance Y diffuses in the direction of the arrows. If a sufficient quantity of solid X is present, then the equilibrium $X + L_e$ is formed at the left side of the membrane; at the other side of the membrane is formed a liquid, which is represented by a point of the isotonic curve eh . If too little solid X is present then is formed at the left side a solution, represented by a point between e and c .

The result of the diffusion is dependent on the ratio of the quantities of the different phases; if we take f.i. the osmotic system, mentioned already in (19):

$$X \leftarrow Y + L_q \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

This can pass f.i. into one of the osmotic equilibria:

$$L_c \mid Y + L_q \quad X + L_e \mid L'_q \quad L_p \mid L_g \quad . \quad . \quad . \quad (21)$$

in which L'_q is a liquid, represented by the point of intersection of the line Yq with the isotonic curve eh . In the first one of those equilibria the solid substance X has disappeared, therefore; in the second one the solid substance Y and in the last one as well X as Y .

The isotonic curves ak and eh divide the field of the unsaturated solutions into three parts, which behave differently with respect to the solid substances H and X . It follows from our previous considerations:

If we bring in osmotic contact the solid hydrate H or the solid substance X :

- a. with a liquid of the field $akcb$, then the hydrate H remains unchanged, but the substance X flows away totally or partly;
 b. with a liquid of the field $akeh$, then as well the hydrate H as the substance X flows away;

c. with a liquid of the field $ehWd$, then the hydrate H flows away totally or partly, but the solid substance X remains unchanged.

The three cases, mentioned above, occur only then, however, when the isotonic curve ak , starting in fig. 1 from point a , terminates in a point k between c and e and, therefore, does not intersect the saturation-curve de . We now imagine the point e anywhere between k and g ; then the isotonic curve starting from a , intersects curve de in a point, which we shall call a_1 . The isotonic curve starting from point e , will also then intersect the saturation-curve ab ; this point of intersection, situated between a and f , is called e_1 . We now find:

If we bring in osmotic contact the solid hydrate H or the solid substance X :

a' . with a liquid of the field $bcee_1$ then the hydrate remains unchanged, but the substance X flows away totally or partly;

b' . with a liquid of the field ee_1aa_1 then as well the hydrate as the solid substance X remain unchanged;

c' . with a liquid of the field aa_1dW , then the hydrate flows away totally or partly, but the solid substance X remains unchanged.

The field, mentioned above sub b , in which as well the hydrate as the solid substance X flow away, is replaced now by the field mentioned sub b' , in which as well the hydrate as the substance X remain unchanged.

The transition of field b into b' occurs when point e in fig. 1 coincides with point k ; the two isotonic curves ak and he then coincide also. In this special case an osmotic equilibrium:

$$E = H + L_a \mid X + L_e \dots \dots \dots (22)$$

can exist also. This is in accordance with the membrane-phase-rule. (Comm. VII and VIII). If we take constant the pressure of the two separate systems, an osmotic equilibrium has

$$n_1 + n_2 - (r_1 + r_2) + 1 - d$$

freedoms. The number of diffusing substances d is here: one; the number of components on each side of the membrane is two and also the number of phases. Consequently the number of freedoms is:

$$2 + 2 - (2 + 2) + 1 - 1 = 0.$$

Therefore, system (22) is invariant, consequently it exists only at a definite temperature and the composition of the two liquids is completely defined.

As is apparent from the situation of the saturation-curves bc and de in fig. 1 with respect to one another, we have assumed that the temperature, for which fig. 1 is valid, is higher than the eutectic temperature of the binary system XY . If we lower the temperature till below the

eutectic and below the melting-point of ice, then we may obtain a diagram like fig. 2. Herein fg represents the ice-curve viz. the liquids saturated with ice, the O.Y.A. of the liquids of this curve increases in the direction of the arrows viz. from f to g .

We now find for this fig. 2:

the O.Y.A. of solid Y is equal to that of the liquids of the saturation-curve bc ;

the O.Y.A. of solid X is equal to that of solid $X + Y$, consequently also equal to that of the liquids of the saturation-curve bc ;

the O.Y.A. of the solid hydrate H is equal to that of the liquids of the isotonic curve al ;

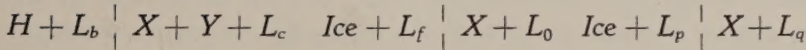
the O.Y.A. of ice is equal to that of the liquids of the isotonic curve fo .

the O.Y.A. of solid Y and of solid X is, therefore, smaller than that of all liquids of the field $abcdgf$.

It is apparent from the position of the isotonic curves in fig. 2 that a.o. the osmotic equilibria:

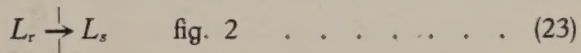


fig. 2



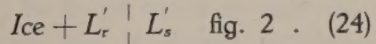
may occur now, which cannot exist in fig. 1:

We now take the osmotic system:



in which L_s has a greater O.Y.A. than L_r , so that the substance Y in (23) diffuses in the direction of the arrow. Again it now depends on the

ratio of the quantities of both the liquids, which will be the result of the diffusion. Two unsaturated liquids viz. can be formed, but (23) can pass also into the osmotic equilibrium:



Herein L'_r is a liquid of the ice-curve, situated between the points f and p ; L'_s is the point of intersection of the line Ys with the isotonic curve, which goes through the liquid L'_r .

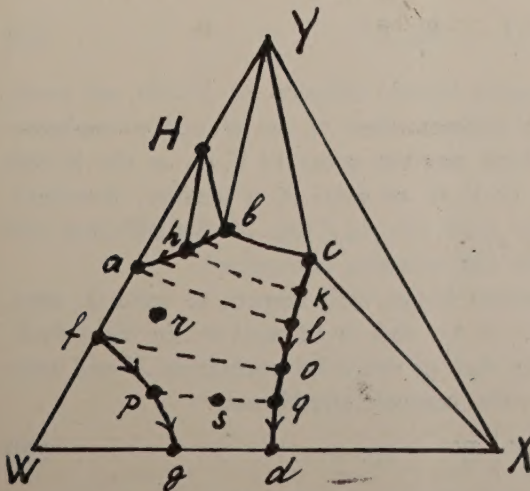
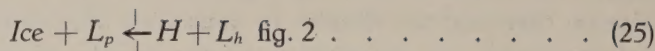
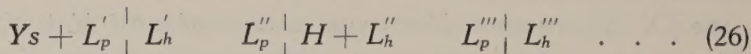


Fig. 2.

In the osmotic system:

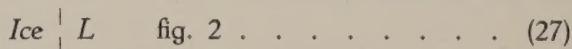


L_p has a greater O.Y.A. than L_h ; consequently the substance Y diffuses in the direction of the arrow. According to the composition of the liquids and the ratio of the quantities of the phases, there can arise from (25) a.o. the following osmotic equilibria:



In the first one of those three equilibria L'_p is represented by a point p' of the ice-curve between p and f ; L'_h is represented by a point h' viz. the point of intersection of the line YK , in which K represents the complex $H + L_h$, with the isotonic curve, starting from point p' . In a similar way we find the position of the points, which represent the liquids L''_p, L''_h etc.

In the osmotic system:

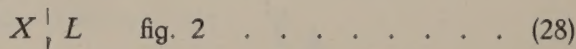


it is dependent on the composition of the liquid L , whether diffusion of the substance Y will take place or not. The O.Y.A. of the ice, as we have seen above, is equal to that of the liquid L_f , therefore, equal to that of the liquids of the isotonic curve fo .

Consequently if we take in (27) a liquid of the field $focba$, then the ice will melt totally or partly with formation of an unsaturated solution, consisting of water + Y or the solution L_f saturated with ice.

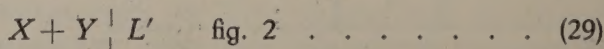
If, however, the liquid L in (27), is situated within the region $fodg$, then no Y diffuses towards the ice and the system rests unchanged, therefore.

If we bring in osmotic contact solid X with a liquid L , then we have the osmotic system:



As long as L is an unsaturated liquid, diffusion of Y can not occur in this system. If viz. a little Y diffuses, then at the left of the membrane the system $X + Y$ occurs, which has the same O.Y.A. as the liquids of saturation-curve bc . As the O.Y.A. of solid X is smaller, therefore, than that of every liquid of the field $abcdgf$ (fig. 2) this diffusion can not occur, therefore, and system (28) remains unchanged.

If L in (28) is a supersaturated liquid with respect to solid Y then, however, it is quite different. If f.i. this is situated in the field bcY , then its O.Y.A. is smaller than that of the solid substance X and consequently (28) is converted into the osmotic equilibrium:



in which L' represents a saturated solution of curve bc .

Summarising some of the previous considerations, we may say:

Ice in osmotic contact with a liquid

of the field $fodg$ remains unchanged;

of the field $focba$ flows away totally or partly.

The hydrate H in osmotic contact with a liquid

of the field $alc b$ remains unchanged;

of the field $aldgf$ flows away totally or partly.

Solid Y in osmotic contact with every liquid of the total unsaturated field flows away totally or partly.

Solid X in osmotic contact with every liquid of the total unsaturated field remains unchanged.

(*To be continued.*)

Physics. — "*The Equilibrium in the Capillary Layer*". By Prof. J. D. VAN DER WAALS Jr.

(Communicated at the meeting of October 31, 1925).

In his theory of the capillary phenomena published in 1893¹⁾ and also still in the "*Lehrbuch der Thermodynamik*" by VAN DER WAALS—KOHNSTAMM, which appeared in 1908, my father has developed a theory of capillarity based on the supposition of a continuous transition from the liquid to the vapour density.

For this purpose he calculated the free energy of a quantity of substance which is in the transition layer between the two phases. With the method followed by him the knowledge of the entropy in the capillary layer was required, and about this he introduced the supposition that it would exclusively depend on temperature and density at the point itself and would be independent of $\frac{dn}{dz}$ or higher derivatives of n (n = number of molecules per ccm. and z = direction normal to the liquid level). In other words: he assigned to every quantity of liquid the entropy which it would possess, if it was amidst a homogeneous phase of its density.

ORNSTEIN²⁾ has pointed out that this supposition cannot be rigidly accurate. He assumed, however, that the radius of the sphere of attraction, within which two molecules still attract each other appreciably, is very great compared with the radius of the molecule. On the ground of this supposition he neglected the influence which the derivatives of the density can have on the entropy.

As, however, the sphere of attraction must not be put much larger than the molecule itself, or at least their radii are of the same order of magnitude, it seems to me that the said influence should be taken into account in a theory of capillarity.

If, however, it is the intention to find the condition of equilibrium of the capillary layer, this may be reached by a shorter way, in which the entropy is not explicitly introduced. We may start from the so-called law of the distribution in space of BOLTZMANN, which teaches that:

$$\frac{N}{v_b} e^{\frac{\epsilon}{kT}} = \text{constant in space}$$

(N = number of molecules per molecular quantity, v_b = the available space in a volume containing a molecular quantity, ϵ the potential energy of a

¹⁾ Cf. among other things J. D. VAN DER WAALS. *Zeitschr. f. Phys. Chem.* **13**, 657, 1894.

²⁾ L. S. ORNSTEIN. *These Proceedings* 11, 526, 1908.

molecule at the point considered, kT = double the mean kinetic energy per degree of freedom at the given temperature). Or, what comes to the same thing, we may start from GIBBS's principle that the thermodynamic potential μ is constant in space, in which μ may be put equal to $RT \times$ the logarithm of BOLTZMANN's expression ¹⁾.

It is known, that we may put in a homogeneous phase:

$$v_b = \{v - 2b(n)\},$$

in which for $n=0$ $b(n)$ approaches to half the joint volume of the distance spheres, and is smaller for greater value of n . In the capillary layer, however,

$$v_b = \left\{ v - 2b(n) + k_1 \frac{\partial n}{\partial z} + k_2 \frac{\partial^2 n}{\partial z^2} + \dots \right\}$$

must be put, in which k_1 and k_2 represent constants. It is the influence of these constants that has been so far neglected in the theory of capillarity.

If the partial overlapping of the distance spheres should be disregarded (in which case it would also be allowed to put $b(n) = b_\infty = \text{constant}$) these constants could be easily calculated. In the first place it is easy to see that then k_1 is zero, at least if also the grouping of the molecules in consequence of the attraction should be disregarded. In reality k_1 will not be rigorously zero. Here it may be pointed out that in the usual calculation of the energy in the capillary layer (see VAN DER WAALS loc. cit., VAN DER WAALS—KOHNSTAMM, ORNSTEIN loc. cit. and others)

also a term with $\frac{\partial n}{\partial z}$ is omitted, which if the "mutual exclusion of the molecules out of their distance spheres" and the group-formation in consequence of the attraction is fully taken into account, would probably appear not to be rigorously zero either.

In the second place it is easy to calculate what part of a layer of the thickness of dz (dz = small compared with the radius of the distance sphere σ , and a fortiori small compared with the whole thickness of the capillary layer) is occupied by distance spheres of molecules which lie on either side of this layer at a distance $< \sigma$. Neglecting the points mentioned above we find for this fraction:

$$\frac{2b}{v} + \frac{b\sigma^2}{5} \frac{\partial^2 1/v}{\partial z^2}$$

so that:

$$v_b = v \left(1 - \frac{2b(n)}{v} - \frac{b\sigma^2}{5} \frac{\partial^2 1/v}{\partial z^2} \right).$$

With the value of the energy calculated before this gives (see inter alia VAN DER WAALS—KOHNSTAMM p. 229 et seq.):

¹⁾ My father calls the quantity in question (loc. cit.) "total potential" and considers a part of it as "thermodynamic potential". It seems, however, simpler to me not to make the distinction of the energy into "internal or thermodynamic" and "external" (among which latter the capillary energy is reckoned).

$$\frac{2a}{v} - c_2 \frac{\partial^2 1/v}{\partial z^2} + RT \ln \left\{ v - 2b - \frac{v b \sigma^2}{5} \frac{\partial^2 1/v}{\partial z^2} \right\} = \mu = \text{constant in space,}$$

in which

$$c_2 = 2\pi \int_0^\infty u^2 \psi(u) du$$

$\psi(u)$ being defined by:

$$d\psi(u) = -f \Pi(f) df$$

and $\Pi(f)$ representing the potential of the molecular attraction at distance f .

When the b -correction in the equation of state may no more be neglected than the a -correction, and the radius of the sphere of attraction r_a is of the same order of magnitude as σ , the term with $\frac{v b \sigma^2}{5} \frac{\partial^2 1/v}{\partial z^2}$ may no more be neglected than $c_2 \frac{\partial^2 1/v}{\partial z^2}$. This is seen when it is considered that $a = 2\pi \int_0^\infty \psi(u) du$, so that c_2 is of the order of $a r_a^2$.

When the last term of the form under the logarithm sign is small in comparison with the other terms, the following form may be written:

$$\mu = -\frac{2a}{v} + RT \ln \left(v - 2b \right) - \left(c_2 + \frac{v b \sigma^2}{5} \frac{RT}{v - 2b} \right) \frac{\partial^2 1/v}{\partial z^2}. \quad (A)$$

which expression is distinguished from the prevalent one only in this that the coefficient of $\frac{\partial^2 1/v}{\partial z^2}$ is not c_2 , but $c_2 + \frac{v b \sigma^2}{5} \frac{RT}{v - 2b}$. It should, however, be borne in mind that in the other formulae of the theory of capillarity the form in question is not always to be substituted for c_2 in order to take the influence of molecular "exclusion" duly into account. In the calculation of the tangential tensions in the surface layer in consequence of molecular attraction, e.g., as carried out by HULSHOF, no change need be applied, i.e. the formula:

$$p_{\text{tang.}} - p_{\text{norm.}} = \frac{c_2}{2} \left\{ \frac{1}{v} \frac{\partial^2 1/v}{\partial z^2} - \left(\frac{\partial 1/v}{\partial z} \right)^2 \right\}$$

remains unchanged. Only v as function z is now found by solution of (A). It is, therefore, different from what it would be without the term with $\frac{v b \sigma^2}{5} \frac{RT}{v - 2b}$.

In consequence of this another value is found for the capillary constant. Besides, this only holds for that part of the tensions that is determined by attraction. The kinetic pressure, too, will not be quite independent of the derivatives of $1/v$. I have not yet examined how great this amount is, and whether it may be neglected compared with the other terms governing capillarity.

Meteorology. — E. VAN EVERDINGEN : *"The cyclone-like whirlwinds of August 10, 1925."*

(Communicated at the meeting of October 31, 1925).

1. When in the morning of August 11 the papers in this country, from all quarters in the southern and eastern part of the country, received reports of heavy devastations, reminding in their description of the effect of a hurricane or a cyclone, especially in consequence of the reports saying that Borculo had been "entirely destroyed", the Meteorological Institute was asked for information about the cause of the disaster and publication of its observations. In both respects we were obliged to disappoint the applicants. We could say nothing about the cause but that it ought to be connected with the passage of the line squall of a thunderstorm, which had attracted our attention also at De Bilt by the darkness preceding it, no particularly severe phenomena ensuing however. Hence local observations did not give any clue, nor those at the four principal stations from which we get telegraphic reports at regular intervals — at the moment of the disaster at Borculo, about 7 o'clock p.m., the weather map shows a shallow depression-centre over the northern part of our country, with lowest readings just below 755 mm., and the maximum gradient of pressure in the triangles Flushing—De Bilt—Maastricht and De Bilt—Maastricht—Groningen remains short of 2 mm. per degree for very divergent directions. It is in good harmony with this fact that at none of the principal stations windforce of any extraordinary importance occurred, at some even the gale-limit was by far not attained. For these reasons, however severe the destructions may have been locally, it is not right to talk of a hurricane or a cyclone — with those words we indicate in meteorology phenomena of a much greater size, with a diameter of many hundreds and a length of path of thousands of kilometers. For the local whirlwinds which often, albeit usually in a much more modest form, accompany thunderstorms, the real dutch name "windhoos" (wind-spout) is best adapted. In this case, in view of their exceptional severity and extension, we may talk of cyclone-like whirlwinds.

2. It is probably on account of the very local character of whirlwinds that we are inclined to believe them to be more rare than they really are. In this country, in the period 1888—1913 whirlwinds of more or less importance occurred on the average on 8 days per annum. During the years 1882—1925 the Meteor. Institute got notice of 82 cases, in which at one or more places damage was caused, comparable with what occurred this time on a big scale : trees knocked down or snapped, hay-cocks blown

up and scattered, roofs or even houses damaged. Fig. 1 indicates the localities, where these whirlwinds occurred: much haunted are Sealand and Friesland, remarkably spared the N.-E. part of Groningen and Dutch



Fig. 1.

Flanders. The total number of cases however is still too small to permit us to draw conclusions, though it seems probable that the proximity of the sea or the influence of heated sands and hills favours the formation of whirlwinds. The distribution of these whirlwinds over the various months was :

Jan.	Febr.	March	April	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
1	2	2	3	9	11	20	19	9	2	4	—

and bears much resemblance to that which WEGENER has deduced in his remarkable book "Wind und Wasserhosen in Europa" — here however June has sensibly less, July more.

The particularity of the whirlwinds of August 10 does not in the first instance consist in the character of the destructions, which has been formerly equalled at various places. We mention f.i. August 29, 1916

when in Limburg hundreds of trees were smashed ; by chance also August 29, 1919, when near Voorthuizen once more hundreds of trees fell ; July 17, 1920, when a similar case happened in Twente, finally July 11, 1924, when Delft and Alphen on the Rhine were visited. Moreover it is possible that local circumstances have played their part, and hence a comparison is difficult. Extraordinary without doubt was however the extension of the region, where the destruction took place. Immediately after the disaster the Meteorological Institute has asked all its voluntary observers in the southerly, central and easterly part of the country to make a sketch of the extension of the damage in their vicinity, and also the principal papers printed such a request. Many persons have answered to this request with much diligence : moreover we were favoured by the cooperation of Dr. H. K. DE HAAS from Rotterdam, who passed his summer-holidays at Barchem and from there recorded the direction in which trees had fallen at many spots over a large area — we will treat of this afterwards. The writer, together with Dr. C. SCHOUTE, adjunct-Director, and Mr. W. WOLTERS, secretary of the Institute, who likewise passed his holidays at Barchem, visited a large portion of the damaged district in Gelderland and Overijssel by means of a motor car, put kindly at our disposition by the municipality of Borculo. Also Messrs. K. ZWART, retired head-teacher, and J. TH. A. BOTH, functionary of the El. C^y. "de Berkel", earned great merit by surveying and charting the direction of fall of trees in the municipality of Ruurlo. Finally we received a map of the whole region visited on two motor car trips in N.-Brabant and Gelderland and Overijssel by Mr. J. G. LEPPER, engineer at Aerdenhout.

The whole of these data enabled us to draw a concise map of the devastated regions. Whereas for some whirlwinds and tornado's, occurred abroad, it has been possible to find out a track, along which within a certain width almost everything is destructed, so much so that f.i. in Scandinavia people speak of an "Asgardsroad" through a wood, in this case one is struck rather by the great lack of regularity and the saltatory character of the destructive action. In the midst of an otherwise uninjured wood we find spots, where all trees have been felled ; along a road of more than 10 km. we find at irregular distances portions, where everything has been devastated by forces across the road, next to large regions almost uninjured — at other places the devastation is limited to a single narrow strip. Therefore it was impossible to map completely the extension of the damage ; hence we have only indicated in fig. 2 the devastated regions by a scale of three degrees : narrow horizontal shadings for the regions, where heavy damage was caused to trees, crossed shadings where also buildings were damaged, black where buildings were entirely destructed. A wide horizontal shading indicates damage at few isolated spots.

On investigation it appeared that no reports of damage in Belgium had reached the Meteorological Observatory at Uccle. On the contrary in Germany damage was caused at a rather large number of places. Though

we did not yet receive complete information¹⁾, we could conclude from reports in the papers that in the N.-E. part the spots devastated are much further apart in the transverse direction than in our country, as may appear



Fig. 2.

¹⁾ While correcting the proofprint we received a map from the „Deutsche Seewarte“ at Hamburg, from which it appears that the squall-lines of the thunderstorm may be followed up to Kiel, and windforce 9—12 of the Beaufort scale was reported largely from three parallel tracks, the prolongation of which to our country contains all the regions devastated there.

from fig. 3. Taking everything together it appears that the length of the track is of the same order as the 4 longest tracks mentioned by WEGENER.

3. It is in the first place the capricious character of the destructions which forces us to ascribe them to whirlwinds. If the effects ought to be

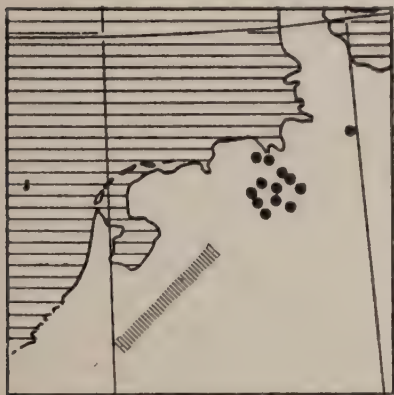


Fig. 3.

ascribed to nothing else but a gale, then the force of a hurricane of such strength would be required, that over a wide area *everything* would have been destroyed. In whirlwinds indeed exceptionally high windforces occur locally and temporarily; even 100 m. p. s. has been mentioned, though nobody can tell with certainty that such velocities have been reached. Indeed, all calculations of the force of the wind from the pressure of the wind, estimated from its destructive effect, yield too high values because it is certain, that other forces of the same

order of magnitude must have been present at the same time. These are the differences in atmospheric pressure, which in the rare cases where a whirlwind came across a barograph have proved to be able to reach 20 to 30 mm. The mean diameter of a whirlwind is something of the order of 100 m. Hence mean pressure gradients of the order of 1 mm of mercury in 5 to 3 meters play a part, locally and near the axis perhaps 10 times bigger. This causes forces of the order of 25 to 40 kg. per m² of an object of 1 m. thickness, hence forces, already equal to the windforces experienced in heavy gales, but differing from these in this respect, that they increase with the thickness of the object on which they are displayed. If such a whirlwind progresses with a big velocity — in this case there is reason to estimate that velocity at 20 to 30 m. p. s. — then there is certainly no time to develop everywhere an equal distribution of pressure, and we may expect quasi-explosive effects of the not expanded air, which is present inside buildings.

That is why roofconstructions are tilted up, roofcoverings are blown off, windowpanes and even walls are thrown outwards at the side opposite to that, exposed to the strongest wind, even if the walls on that side remained intact, as has been observed in many cases. This also explains the possibility of very different directions of fall of trees at neighbouring places. This, lastly, explains why heavy objects may be carried through the air over rather large distances, borne by a diminution of pressure over and before them, and forced on by air streaming towards the depression and ascending at the same time ¹⁾).

¹⁾ I consider this explanation to be more acceptable than WEGENER's supposition, that transport over large distances would take place in a horizontal part of the vortex.

However — even when taking into account these forces, wind pressures of the order of 200 kg. are required to explain several of the masterpieces accomplished by whirlwinds; hence we are obliged to assume wind velocities of more than 50 m. p. s.

We give in the following the few observations, where the funnels of the spouts were actually seen.

Borculo. Dr. J. W. GRONDIJS. "From the Southwest approaches a funnel of a dirty yellow-greenish colour. In a short time the colossus has reached the village. Violent wind. I see the trees in the Bloemerstraat falling with a single blow, direction S.—N., but at the upper end of the street the trees lie N.—W.—S.—E."

Borculo. H. W. HEUVEL. "The clouds arrived revolving like a whirlpool. Somebody tells: At once we saw in the sky something strange. A long straight tube, which rotated quickly at a horrific pace and approached with a terrifying roar."

Nijmegen. R. TEN KATE. "When the squall had approached a good deal and it had grown very dark, I saw at once a large frayed cloud approaching somehow from N.—W. with an amazing velocity in the direction of the squall. This attracted my attention to such a degree, that I inspected the sky more closely and then I saw from a southern direction another cloud approaching in the direction of the squall. Just before the squall had reached the North-Southline over my dwelling, it was reached by the two clouds, whereupon they assumed together a rotary motion and proceeded in the direction of the squall. At the moment this rotation started, the trees began to wave wildly without a distinct direction. The rotation was rather quick (one revolution in about 6 seconds.)"

Zevenaar. Drs. J. G. A. HONIG. "Various persons have seen a dark rotating(?) column moving from S.—W. to S.—E."

Uden. J. TH. FRUNT. "The clouds came together from the directions N.—W. and S. Everyone cried fire. But there was no fire. Then the clouds descended to about 50 m. above the surface, and much sand came with them then towards Uden, there the whole motion of the air was involved in rotation."

These data and the ensuing notice about the estate "Het Espelo" prove beyond doubt that whirlwinds have occurred. The very dark sky and the velocity with which everything proceeded are sufficient reasons, why more numerous descriptions were not received.

4. Details of local destructions are to be found at random in papers and periodicals, and this is not the occasion to treat further of these. Very important however is the result of Dr. H. K. DE HAAS' investigation, which is resumed in fig. 4. For orientation the railways and principal roads in the vicinity of Ruurlo, Lochem and Borculo have been indicated. Every arrow marks the direction of fall of a tree, which stood at the spot, indicated by the *point* of the arrow. Some curved arrows indicate spots, where one

gets very strongly the impression of quickly rotating forces, f.i. because the branches of a group of firs were entirely coiled up. Only those directions



Fig. 4.

of fall have been inserted, where it was certain that the tree lay untouched. As of course in many spots soon clearing had taken place, the number of arrows and their distribution do not give a true picture of the intensity of the whirlwinds; but they do, as far as the direction of the largest forces in any point is concerned. Fig. 5 is a reproduction of the sketch received from Ruurlo. Here the directions of fall diverge still more.

A comparison of these figures with tracks of whirlwinds in WEGENER's collection shows, that this case is among the most complicated. If a single whirlwind proceeds along a straight line, we are able to predict entirely the direction of fall with respect to the track, after making certain suppositions about the velocity of translation and the angle of incidence of the wind with respect to the axis of rotation. It then appears that near the centre all trees must fall in the direction of propagation, whereas on both

Nothing like such a picture here — only small portions along certain roads show some resemblance to it ¹⁾, so as to favour the supposition, that a great number of whirlwinds of relatively small dimensions has been in action, and that sometimes, diverging from the principal direction, these whirlwinds followed the rows of trees along the roads. It is even not impossible, that some whirlwinds followed an adverse track, or showed an adverse direction of rotation. At various places it was possible to conclude to the time-sequence of the successive hurricane blasts, because we can take for granted that the tree lying uppermost had fallen latest. On the evidence of facts like these Dr. DE HAAS thinks he must assume that N.-W. of Borculo several right-hand rotating whirls have occurred.

In the Ruurlo map we are struck especially by the great number of trees, felled from N.-E. in the centre of the Gr. Meene, secondly the great number of isolated spots with numerous felled trees amidst uninjured regions. In these smaller spots the direction of fall does not diverge so much. Everything seems to indicate, that the whirlwinds, descending from the cloudlevel, touched the surface here and there in a saltatory way, and on the occasion of an extremely deep descent over the Gr. Meene developed enough force to fell trees also at the front side. If we assume a general motion in the direction N.-E., we have to distinguish at least 5 whirlwinds in this case only. In these considerations we start from the supposition, confirmed by numerous observations in the case of solitary whirlwinds, that next to the surface the rotation plays only a secondary part in the whirl, the afflux of air being the principal phenomenon. Hence the origin of the rotation, which constitutes the cause of the diminution of pressure, must be looked for in higher strata. In this supposition the progressive motion of the whole mass of air is added to the afflux behind the whirl near the track of the centre, subtracted from it on the front side. If we put the general windvelocity at 20 m. p. s., the felling of trees on the front of the whirl would indicate a velocity of afflux of 40 to 50 m. p. s., if we overlook the pressure-forces, — at the backside then velocities of 60 to 70 m. p. s. must have occurred. We remarked before that the action of pressure differences weakens these conclusions, except in the case of tall trees like firs or poplars, where the pressure effect cannot be very large. But also in other ways estimates are obtained of windforces between 50 and 80 m.p.s.

5. In spite of the very extensive investigations on whirlwinds we cannot say that at the present moment their origin is completely explained. Many facts and experiments however are in favour of the mechanic theory, which assumes that a horizontal vortex is formed when an ascending current forces its way into layers of air, where the velocity strongly increases

¹⁾ On the estate „Het Espelo" near Enschedé f.i. in a wood a passage has been cut, about 10 m. wide, in which the trees not only have been blown down, but also snapped off. (J. VERKOREN Jr. Director of the Gas- and Waterworks, Enschedé).

upwards. Considering that in thunderstorms firstly strong ascending currents occur, but a thunderstorm of some importance requires also an inversion layer, over which almost always a sudden change in the wind may be expected, this explains at the same time the frequent simultaneous occurrence of thunderstorms and whirlwinds.

Starting from this point of view we have tried in the first place to coordinate the facts, observed at the surface, on the supposition that the whirlwinds occurred everywhere at the moment of the passage of the principal line-squall of the thunderstorm. A great number of reports of the thunderstorm were available for determining the motion of this squall line. Though some possibility of confusion arose from the occurrence of several other thunderstorms, f.i. at de Bilt half an hour before the principal squall, we are of opinion that the squall-lines copied on fig. 2 have a sufficient degree of certainty ¹⁾).

A first item of the research dealt with the air-pressure at the surface. In consequence of our request in the papers for communications of observations and sketches for this day, we got a.o. barograms from Nijmegen, Berg en Dal, Lochem, Goor, and Boekelo, all localities situated pretty near the track of the whirlwinds.

None of these diagrams did however show extraordinary variations of pressure, and this is what we might expect, seeing that the distance remained much greater than the diameter of the biggest whirls. Very important was therefore the receipt of the diagrams of the registering gas-pressure gauges at Nijmegen from the Director of the Gas-works, Mr. G. Philips, as the gauge at Hatert was only few kilometers from the track of the whirlwinds, and did show much bigger variations. In order to reduce the indications of these watermanometers to variations of air pressure we assumed, that a barograph, stationed at Nijmegen on the Oranjesingel, would have registered the same variations of air pressure as a barograph, installed at the Gas-factory, save a difference of few minutes in time. After correction for the difference of level of the observing stations and reduction to pressure in mm. of mercury, the differences of the watermanometer at the factory and at the other points allowed the calculation of the air pressure variations there. We know that all registering cylinders show errors of time, and in this case these were very considerable. Therefore we assumed, that the most marked increase of pressure, shown in the diagram, coincided with the moment of the passage of the whirlwind, so as to allow a calculation of the time-error from the time of passage of the squall-line. The same procedure has been adopted afterwards for the barograms in the vicinity of Borculo.

¹⁾ The very large velocity of this squall, about 72 km. per hour explains the velocity with which, according to eye-witnesses, the destruction approached, and is in harmony with the estimates of the lapse of time in which Borculo was destroyed, which vary from 7 minutes to a quarter of an hour. (Addition during correction: In Germany the velocity increased even to 90 km. per hour).

Fig. 6 shows the barograms for Hatert and Daalsche Dijk near Nijmegen, obtained in this way and compared with the diagram of the float-mercury-barograph at de Bilt. The latter shows a marked thunderstorm-unrest, but nothing particular at the passing of the line-squall at 6.00 — the diagrams for Nijmegen however show very large fluctuations, especially at Hatert.

As evident from fig. 2 the destruction stopped temporarily just beyond

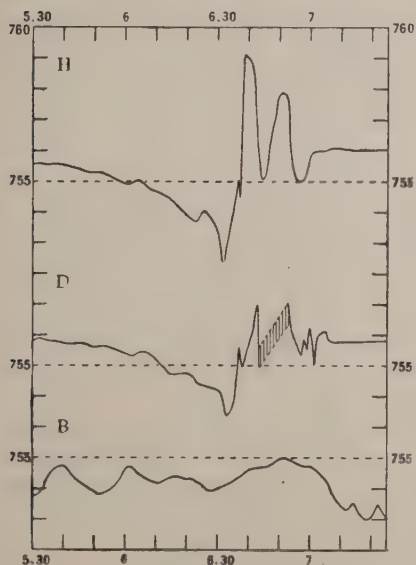


Fig. 6.

Nijmegen, and therefore it was not certain, that the difference in registration at Hatert and Daalsche Dijk was only due to a difference in distance from the track of the whirlwinds. For this reason we abstained from trying to estimate the fall of pressure on the central line by an extrapolation.

Using the observations at principal stations, second order stations and a number of other barograms received, we have then drawn the diagrams in fig. 7 for the distribution of pressure at 2, 4, 5, 6, 6.30 and 7.00 p. m. Only at the moment of the passing of the whirlwinds extraordinary isobars are displayed, but these are very remarkable indeed, as in the neighbourhood of the

track of the whirlwinds, they indicate pressure gradients up to 55 mm. per degree, which, if occurring over larger areas and longer periods, would cause hurricanes of the most destructive kind.

With isobars of a curvature as here occurs however by far the greater part of the gradient is required for change of direction, and moreover it is evident that this pressure distribution did not persist long enough to develop its full windforce. There are no indications of the occurrence of similar gradients farther S.-W., before the whirlwinds appeared. Therefore though we don't exclude the possibility, that in the first place N.-W. ly gales and whirlwinds with a southerly track may be explained by these pressure gradients, we are of opinion, that on the whole these phenomena are to be regarded rather as sequences of than as causes for the formation of the whirlwinds. A large part of the rise in pressure can be ascribed to the replacing of warm by cold air: a temperature difference of 6° C. through a column of air 2000 m. in height gives a rise of pressure of about 4 mm.

6. Fig. 8 shows the distribution of the rainfall in our country according to the reading of the raingauges, at 8.00 a.m. on August 11. It would have been desirable to draw this map exclusively for the rain that fell

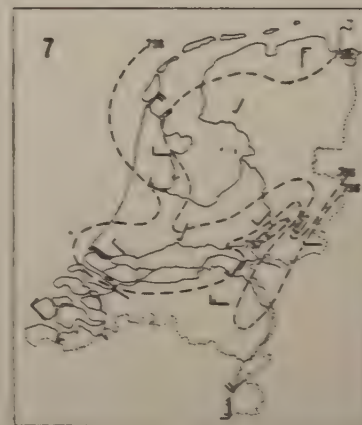
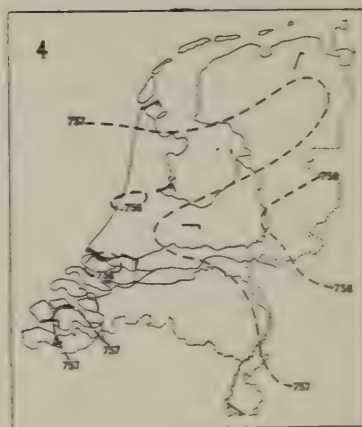


Fig. 7.

with the squall charted in fig. 2. But only a small fraction of the observers has measured separately the rain of this squall, and we only know that at various places this squall has indeed had the largest share in the day's rain. As main result of this investigation we find: 1^o. that the quantity of rain has been very different, varying between 38.5 mm. and nihil; 2^o. that the region devastated by the whirlwinds has not experienced the heaviest precipitation, but that a region of heavy rain is to be found somewhat to the North and to the left of the track of the whirlwinds in Brabant and Gelderland.

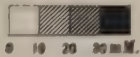


Fig. 8.

From the intensity of the rain the velocity of the ascending currents can be calculated, if we know the vertical extension of the saturated layer and its temperature. If we put the temperature at the surface at 25° C., the weight of the layer of air per m². at 1000 kg., then for a rainfall of 35 mm. in half an hour a vertical velocity of more than 6 m. p. s. is required. Hence it is very probable, that locally and temporarily the ascending velocity surpassed the limit of 8 m. p. s. and so grew larger

then the highest velocity of falling raindrops. Perhaps this is the explanation for the phenomenon, that at many places the squall was preceded by an inkblack sky, from which rain did not yet fall. If such ascending velocities occur simultaneously over a few square kilometers, and if they are fed at the front part of a wedge of incident cold air only by an afflux with a height of a few hundred meters, then we find even in the case of a supply from all sides velocities of more than 20 m. p. s., with an afflux from one side much higher values.

7. The direct cause of the thunderstorm was given by an invasion of cold air from W. under a warm and moist body of air supplied from South. Neither the temperature which occurred even in the eastern part of the country, which remained below 30° C., nor the fall of temperature can be called extraordinary, and doubtless these figures have been surpassed considerably in many heat-thunderstorms. It will appear later on that in the higher strata more important differences were found. Figures 9 and 10 give the course of the isotherms at 2.00 and 7.00 p. m., the arrows indicate roughly the streamlines of the two masses of air. Given the irregular character of the pressure variations and the variable winds caused by these, many more observations would be required to completely determine these lines. Even in this rough form however they suggest the rising

currents due in the Achterhoek (Eastern Gelderland) about 7.00 p. m. and the ascending currents on the Veluwe (Western Gelderland), which fed the heavy winds in the back of the squalls.

It is evident from a comparison of fig. 2 and 8, that at different points

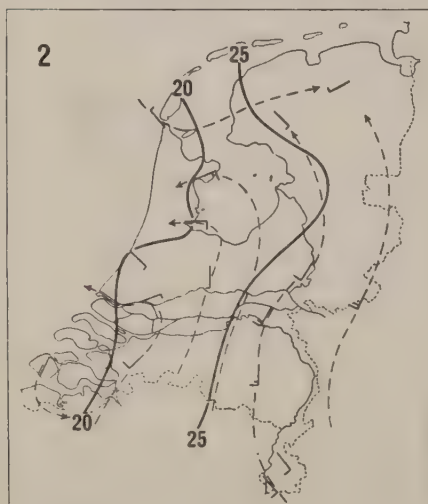


Fig. 9.

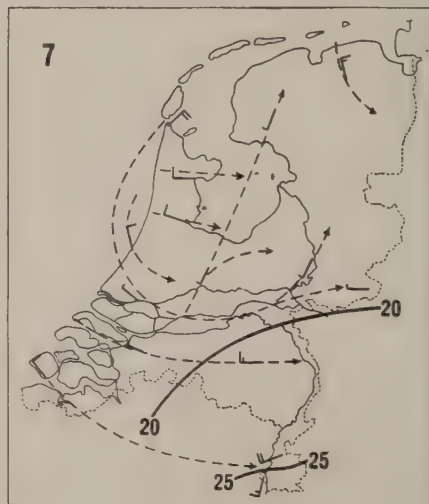


Fig. 10.

of a same squall-line very different amounts of rain fell; also fig. 10 suggests various independent squalls, so as to permit doubt, whether the squall-lines connected by a dotted line in the North and the South of the country really belong together. Hence the conditions of flux differed much locally; other differences in the ascending velocities depend on the orographic conditions, whereas preceding thunderstorms may have altered here and there the conditions in the higher strata. Together these are sufficient reasons to explain the large differences in rainfall.

8. In consequence of the constantly increasing application of the investigation of the upper air to the daily forecasts of the weather, we have at our disposal rather complete material of temperatures and velocities of the air in the lower kilometers. By means of aeroplanes the temperature was determined up to a considerable height at Duxford in England, Uccle in Belgium, Helder and Soesterberg in our country, by means of kites or cable balloons at Lindenberg and Friedrichshafen in Germany. Moreover, observations from mountain stations were available in Germany, Switzerland and France. Numerous observations of pilot balloons give the wind directions in the higher strata at various times of the day. Fig. 11 and 12 give a compilation of the morning observations for the levels of 1000 and 2000 m. (1 m.m. = 2.5 m. p. s.)

Remarkable is here especially the large temperature difference at the level of 1000 and 2000 m., between England and Central Europe. With a view to the observation on the Puy de Dome in France, which gives a temperature of 20° at 1500 m., we may assume that this condition existed

likewise in Central France and that the southerly to southwesterly winds at 1000 to 2000 m. carried this warm air to our country. Above 2000 m.

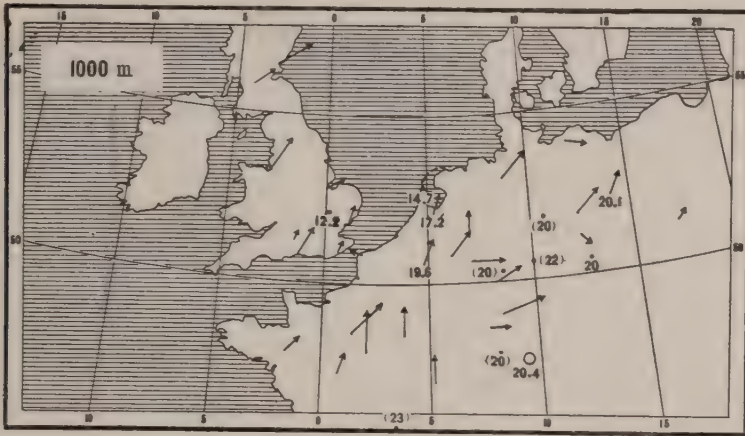


Fig. 11.



Fig. 12.

the temperature over Friedrichshafen decreased quickly. Whereas in the morning the lapse-rate of temperature over Soesterberg was not very favourable for developing a thunderstorm, conditions grew more and more favourable for such a storm in the afternoon, the more so in the easterly part of the country, where the temperature at the surface was much higher, and surpassed 29° C. If we assume that the conditions experienced in the morning over Friedrichshafen had spread in the afternoon from France to our country, then we find a total lapse of 18° up to 3000 m., in which however at about 2000 m. an inversion must have occurred. These are conditions for heavy thunderstorms as soon as other causes force the lower strata upwards and make them break through the inversion. These other causes are to be found in the much lower temperature and more westerly direction of the air layers in the West, which penetrated like a wedge under the warmer and moist air.

As first demonstrated by MARGULES, the gravitational force accomplishes a considerable amount of work when cold masses of air, which first were found beside warmer masses, spread out under these. With a vertical extension of the layers of 3000 m. and a difference of temperature of 5° C. the whole mass of air may obtain a velocity of 12 m. p. s. The sudden outbreak of stormy winds during a thunderstorm may completely be explained in this way — but velocities of 50 m. p. s. are not easily reached in this manner — for these we want the cooperation of a heavy thunderstorm-squall with a whirlwind.

9. We now come to the observations which in our view contain the key for the explanation of the whirlwinds. The older theories, which f.i. connected the whirlwinds with a rapidly rising current of air, which was said to cause vortices, or which ascribed them to the encounter of two strong wind currents of different directions in the same level, have been found to be failing in most cases. Rapidly rising currents often show no rotation at all. Air currents of very different direction and velocity occur in the same level generally only at distances, very large as compared with the diameter of a whirlwind. On the other hand it is shown daily that very different velocities may occur one *over* the other and that in that case, especially when inversions limit the turbulence, very sharp discontinuities may be found. Gradually it has become probable, that these continuities of the wind are necessary for the formation of vortices, and that these vortices therefore will rotate in the first instance about a horizontal axis, the higher parts showing the largest velocity in the direction of propagation. According to hydrodynamics, if friction is not considered, a vortex-thread can end only at the boundary of the fluid. WEGENER and KREBS have made the supposition, that waterspouts and whirlwinds constitute the bend-down ends of the horizontal vortices in the higher strata. If both ends are bent towards the earth, the end on the right of the horizontal vortex will show a cyclonic rotation (anti-clockwise), the end on the left an anticyclonic rotation or right-hand rotation. The effect of the earth's rotation on the masses of air streaming towards the vortex will however favour the cyclonic rotation; moreover the other end may bend upwards and terminate at the inversion layer. As a matter of fact observation points to a preponderance of left-handed rotating whirlwinds. Of course the friction near to the surface will cause all kind of complications — f.i. it appears that often a division into a number of whirlwinds occurs.

If this view is correct, the formation of a whirlwind requires a discontinuity of the wind in the vertical and a strongly rising current, in which the discontinuity causes a horizontal vortex. We should expect then, that the yearly range of the whirlwinds bears a certain resemblance to that of the thunderstorms, as the latter ask for a strong rising current, breaking through an inversion. It appears that such is really the case; in this

country, the yearly range of the thunderstorms is, expressed as percentages of the number of days with thunderstorms a year :

Jan.	Feb.	March	April	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
2.5	2.5	5.0	8.2	13.2	14.1	15.4	15.1	9.0	8.2	3.9	2.9

that of the whirlwinds :

1.2	2.4	2.4	3.7	11.0	13.4	24.4	23.2	11.0	2.4	4.9	—
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In July and August the whirlwinds are comparatively very numerous.

Also the local differences in frequency of thunderstorms — winter thunderstorms more frequent on the coast, summer thunderstorms on the higher sandy grounds, — are shown in the local distribution of whirlwinds given in fig. 1. If the whirlwind constitutes the vertical prolongation of a horizontal vortex, connected with the rising thundercloud, situated to the left of the whirlwind when looking in the direction of propagation, then the maximum precipitation must fall on the left side of the track of the whirlwind. This is confirmed by our fig. 8. To the right of the track are even places, where no precipitation at all has fallen.

It remains to prove the existence of a considerable discontinuity of the wind. Indeed, in the afternoon of August 10 such a considerable increase of the wind upwards was observed as well at de Bilt as at Uccle, that at first an error was suspected. In calculating the velocities of the various layers of air from the observation of pilot balloons from one station, we have to assume the velocity of ascension to be constant — if the balloon gets a leak, then the velocity of ascension diminishes and too big velocities are calculated. The observation at de Bilt at 1.35 p. m. yielded the following results :

Height	500	1000	1500	2000 m.
Velocity	5	13	20	31 m.p.s.
Direction from 190	200	220	220	220 ¹⁾

The observation at Uccle was made even later, 4.44—4.54, Amsterdam S. T. hence nearer to the disaster

Height	500	1000	1500	1800	2000 m.
Velocity	16	24	36	41	36 m.p.s.
Direction from 218	222	226	224	223	

Lastly it appeared that in the morning in East-England, at Orleans and at Brussels at heights from 2000 to 2400 m. velocities of 20 m. p. s. occurred between masses of air, moving slower as well further W. as further East. Hence there is no doubt, that a very important discontinuity of the wind

¹⁾ In degrees, counted from N. through E.

occurred at a height of 2000 m., leading to velocities of perhaps 40 m. p. s. about the time of the disaster. Such velocities are very rare, especially with a pressure distribution without large gradients at the surface — therefore we may ascribe the serious character of the disaster in the first place to this important discontinuity.

11. The cause of the local occurrence of such high velocities is to be found in the increase of the pressure differences at the level of 2000 m., in consequence of the distribution of temperature, and in the increase of pressure differences at various levels by the secondary depression which passed over our country in the afternoon. The warm masses of air moved from a southerly or south-westerly direction, the cold masses showed a more westerly direction, which accentuated itself especially in the afternoon and penetrated at the surface during the thunderstorm. The difference in barometric pressure between S. England and Friedrichshafen, which in the morning amounted to 7 mm. at 2000 m., to 10 mm. at 5000 m., occurred probably in the afternoon over a distance shorter by one third and was augmented by about 2 mm. by the secondary depression, which must have had its origin at least partly in much higher strata. In this way we arrive almost at doubling the gradient, which hence reached a value of almost 3 mm. per degree, between Brussels and Friedrichshafen even more. This explains sufficiently velocities of such magnitude: the gradient wind at 2000 m. is about 13 m.p.s. per mm. of gradient.

With the general motion of the whole weathersituation towards E. or N.-E., also the region with high windvelocities was transported Eastward. At 11 o'clock the limit just touched the Eastcoast of England (Felixstowe); with the W.-E. components, which during the afternoon were found over England, the displacement until 7 o'clock in the evening would have been about 40, that is about to the central part of our country.

The boundary will probably have had the direction of the warm winds, S.-S.-W. or S.-W., hence in the northern part of our country it was situated further towards East. This is probably the reason, why the thundercloud which caused heavy rain in the northerly part of our country, from Zwolle to Delfzijl, at den Hulst 32.2 mm. in the course of half an hour, has given no rise to the development of heavy whirlwinds, though the destructions in the neighbourhood of Staphorst and Zorgvlied (Smilde), (the isolated spots north of the principal track), show that the limit was nearly reached.

It is evident from the foregoing, that for the formation of so heavy whirlwinds over a large region a rare coincidence of circumstances is required. We need not be amazed therefore, that in order to find in our country a disaster, comparable with that of 1925, we have to go back to the very heavy whirlwinds of August 1, 1674, which made collapse a number of church towers and part of the cathedral at Utrecht.

Résumé : 1. The destructions on August 10, 1925 are to be ascribed to heavy whirlwinds, accompanying a thunderstorm otherwise of no extraordinary intensity, and distinguished from the whirlwinds observed during the last 40 years principally by the extension of their range.

2. Almost all the phenomena observed are in harmony with the view that whirlwinds are the prolongations towards the surface of horizontal vortices in higher strata, a view first pronounced by WEGENER and KREBS.

3. The extraordinary intensity of the whirlwinds is a consequence of the very strong increase of the wind over the level of 2000 m., which in its turn is explained by the very large differences in temperature between W. and Central Europe in the lower kilometers.

with their usual mode of preparation, which may be lowered to 0.13 mg. by a certain process of purification, that DICKENS and DODDS describe the substance as a brown oil, of which in the purest preparation 25 mg. correspond to one rat-unit, whereas STEINACH's mouse-unit weighs 9—13.5 mg. or more. This may be considered pretty well to be the present condition of things.

Therefore we think it progress that we are now able to prepare a substance which brings about very extensive cyclic changes, in a much simpler manner than the rather complicated method of the American and English investigators: moreover, our product has a dry residue of considerably less than 0.1 mg., even less than 0.01 mg. per mouse unit; it is quite free from proteins and is water-soluble. Up to this moment there is no publication containing any proof that the active principle, dissolved in water, is in true solution.

Definition and standardization.

To prevent the repeated use of the term "relatively pure hormone of the estrous cycle" we propose the name "Menformon" for the substance in question, but with the restriction that we understand by it only a substance *which contains at least 10 mouse-units per 1 mg.*

As a mouse-unit (thus included in the definition of "menformon") we define the smallest quantity which is able to produce undeniable cyclic changes of the vaginal epithelium (at least surpassing, at their maximum, the stage described by ALLEN c.s. as "pro-estrus") within 72 hours after the last injection in at least 2 of every three castrated mice, on simultaneous injection.

The injection should be performed with intervals of 4 hours, giving $\frac{1}{3}$ of the total dose at a time; the animals must be castrated at least 25 days before, and they must have been submitted thereafter to a daily control to show that spontaneous cyclic changes are completely absent. In our laboratory the histologic preparations are controlled by two persons independently of each other and without knowledge of what has happened with the animals.

In a few words we shall explain what we mean by "complete absence of cyclic changes". The anatomical investigations by STOCKARD and PAPANICOLAOU¹⁾ furnished a means by which further progress in this field was made possible.

It is well known that the vaginal epithelium in the period of rest of the mucosa consists of 2—4 layers of cells, through which large numbers of leucocytes find their way to the lumen of the vagina. During the estrus the epithelial cells multiply until there are to be seen 14—18 layers of them. The upper layers are cast off as cells and plates without nuclei; the passage of leucocytes is prevented. *This absence of leucocytes is the most*

¹⁾ Am. Jl. Anat. 22, 225, 1917.

important criterion; yet a mere decrease of leucocytes and the massal appearance of epithelial cells which ordinarily are only to be found here and there, constitute an unmistakable change as compared with the negative finding in castrated animals not subjected to the treatment. We consider the action of a substance as *positive* (+) when in the microscopic preparation nearly all leucocytes have disappeared and the epithelial cells predominate, of which last category there must be about as many cells with a nucleus as without one. Frequent occurrence of epithelia *without* nuclei warrants that the pro-estrus has been *passed*, as required in our definition of the mouse-unit. Maximally positive (++) we call the stage in which epithelial cells with nuclei have completely disappeared, and in which moreover most of those without nuclei are cornified.

Up to this moment our experiments have shown that a "maximally positive" (++) reaction (on distributing one mouse-unit over one day) is more often obtained with the original follicular fluid than with the substances prepared from it. These often only give the (++) reaction on repeating the injection the second or third day, even with less than one mouse-unit. This is not merely a question of dosage, for giving 2 or 3 mouse-units in one day needs not produce the same effect. Probably quicker absorption plays a part combined with quicker excretion of the purer preparations, so that only by repeated injections a sufficient concentration during a certain time may be obtained. Thus to obtain a certain effect two factors are concerned: concentration and time.

Important though this problem be, it does not in the least influence the fact that on injecting thrice in the course of one day an unmistakable effect may be obtained, which instead of the continually negative findings in untreated animals doubtlessly shows a change in the sense of the typical estrous cycle, possibly only in a more rapid succession of its stages.

Our diagnoses made in this manner in smears have been confirmed by histologic control of some mice killed for the purpose: when the smear had been labelled (+) their vaginal epithelium showed 10—14 layers.

We require that the animals be castrated at least 25 days before, and that for every experiment at least 3 animals be used, because only the suppression of several cycles seems to make complete castration largely probable; further, because chance can only be excluded when more than one animal, and even at least three, are being used. On using only two animals, different results can compensate each other.

Preparation.

Our first preparations were made from follicular fluid, which, if unchanged, contains in our experience between 600 and 1200 mouse-units per kilogram. ALLEN and DOISY with their method found 2000 rat-units maximally, DICKENS and DODDS c.s. with their extraction-method only 200 rat-units, an amount corresponding to that obtained from ovaria

without follicles. The English investigators therefore decline the particular part ascribed by the Americans to the follicular fluid: we ourselves are inclining toward the American point of view, though we have been able to prepare menformon from whole ovaries and placentae too. Often the follicular fluid, when not quite fresh and sterile, is toxic, so that the animals sometimes die within 1—3 days after the injection. The follicular fluid we sucked from the follicles with a syringe, as far as possible under aseptic precautions.

After observing repeatedly the activity of the unchanged follicular fluid, we tried the purification process of ALLEN and DOISY a few times. This too yielded positive results, though we did not reach the low dry residue as recorded by ALLEN and DOISY in a few cases. Yet we did not spend too much time upon it, because the method is extremely complicated.

We have the impression that, for the first and decisive phase especially, the authors have been strongly influenced by the principles of insulin preparation, and, secondly, by the idea that the active principle would be insoluble in water. This we thought without proof, for though most previous investigators, e.g. FRAENKEL and HERMANN¹⁾ maintained the exclusive lipoid-solubility of the ovarian hormones, others, e.g. L. ADLER already in 1912, mentioned the activity of aqueous extracts. Possibly the insolubility in water was only caused by the fact that in the various methods the active principle is extracted together with water-insoluble substances, which prevent the active hormone from dissolving in water.

Thus we aimed at obtaining a water-soluble product.

We thought it one of the most important problems how to free it from proteins. For this purpose, we tried several well-known methods.

After boiling with dilute acid the remaining dry residue was very high, but this experiment confirmed the *thermostability* of the hormone as shown by ALLEN and DOISY. Precipitating with trichloro-acetic acid gave much less dry residue, the solution was perfectly clear, but showed to be very toxic, even after neutralizing. In control experiments with solutions of the sodium salt of trichloro-acetic acid however (of a concentration corresponding to that used in precipitating the proteins) the animals died in much the same manner (perhaps a chlorine intoxication?).

Removing the proteins by the FOLIN-WU-method (as usual in determining the blood-sugar content) proved unsatisfactory.

Better were the results on using colloidal ferric hydroxide.

To one part of follicular liquid 4 parts of physiological saline and about $1\frac{1}{2}$ parts of a 3 % solution of colloidal ferric hydroxide were added, the mixture was then centrifugalized. This yields a turbid yellow fluid which apart from some colloidal $\text{Fe}(\text{OH})_3$ contains the active principle. On evaporation at low temperatures (35° C.) the yellow turbidity precipitates as minute particles, and there results an almost water-clear liquid, faintly

¹⁾ (German) Patentschrift N^o. 309482. Klasse 12.0. Gruppe 26, Ausgegeben 23. 11. 1918.

opalescent and with a dry residue of about 3.3 % : 0.5, 0.75 and 1 cc. of this liquid, injected into castrated mice, constantly gives positive results (0.5 cc. of this liquid corresponded to 0.3 cc. of the original follicular liquid).

The application of this method to larger quantities of follicular fluid, followed by removal of $\text{Fe}(\text{OH})_3$ by means of evaporation and H_2S , showed its usefulness. Such a solution may for instance show a dry residue of 3.35 %, an ash content of 1.06 % and a nitrogen content of 0.47 %.

From such a larger batch we injected, among others, 9 mice, every animal three times ; three mice got 0.5 cc., three others 0.75 cc. and the remaining three 1.0 cc. All except one showed a positive reaction. Repeating the experiment with 0.5 cc. gave strongly positive reactions. Possibly the limit value was even less. But — when prepared by this method one mouse-unit contained about 50 mgr. of dry residue, which compares very unfavourably with the results of the American method and even with those of the English mode of preparation. The only advantage of our method consists in its simplicity.

In view of this unsatisfactory dry residue we further simplified our technic, combining the precipitation of the proteins with the extraction, and then trying to make the hormone pass over into water again. The leading principle must be : to extract the menformon as completely as possible with the least impurities (lipoids included) possible.

To give an example : 10 cc. of follicular fluid and 10 cc. of chloroform are shaken together. The proteins coagulate in part. Then the liquids are separated, the chloroform is evaporated and the residue is taken up in about 5 cc. of distilled water. A perfectly clear watery solution results, having an unweighable content of solid matter. Its activity is rarely more often less than half that of the original follicular liquid, that is to say, if, c.g. 0.6 cc. of follicular liquid contained one mouse-unit, 0.6 cc. of the aqueous solution again contained one mouse-unit.

Though we have already a rather large number of experiments at our disposal we are not yet prepared to say anything definite about the number of mouse-units which may be obtained in this way, nor do we predict anything about the possibility of obtaining by further purification a larger amount of units per cc. than are present in the original follicular liquid, for instance by removing substances with an opposite action (anti-menformon). To settle these questions the number of experiments must be still much greater than it is at present, and, moreover, the limit of accuracy of the method of standardization must be much better known.

Besides chloroform we used carbon sulphide, carbon tetrachloride, benzene, petroleum-ether, tetralin, ligroin, ether, acetone and ethyl acetate. In principle all these yielded identical results, but our experiments are still too few in number. Often the residue from the volatile solvent at first formed an emulsion with the water : in most cases however a simple

filtration made the fluid clear ; often the solution was water-clear from the first, and then its content of solids was minimal. Probably this depends, besides on the nature of the solvent, (which must dissolve menformon with as little water-soluble impurities as possible) on the freshness of the follicular liquid, and further on the completeness of the precipitation of the proteins, which then take the other colloidal constituents with them. Finally the completeness with which the volatile solvent is removed plays a part, eventually also the temperature at which this has been done. (We worked at atmospheric pressure, in vacuo and in an air current at about 35°).

All these factors, and many more, are still to be investigated much more completely.

Besides shaking out the follicular fluid directly and eventually coagulating the proteins at the same time, we first dried the follicular liquid itself or solutions prepared from it by the $\text{Fe}(\text{OH})_3$ -method mentioned above, then we extracted the dry residue with chloroform, which in its turn was evaporated, after which the residue was dissolved in water. This solution too yielded identical results, with minimal amounts of solids. This leaves no doubt that *menformon is water-soluble*.

Our method may be briefly called the "water-method". When starting from larger quantities, e.g. 1 Litre, of follicular liquid, we found it useful to prevent the forming of emulsions when shaking out with the volatile solvent : this may be done by completing the coagulation of the proteins by adding salt or acids, by centrifugalizing etc.

Solubility, dialysis.

Whether the clear solution obtained by extracting the residue from the volatile solvents with water is a true solution or only a colloidal suspension, we tried to decide by means of dialysis. In previous experiments on purification of our preparations made by means of colloidal ferric hydroxide we found that the activity is lost by dialysis against running water through parchment or collodion membranes. On dialysis of preparations made by the "watermethod" and containing in 10—20 ccm. about 40—80 mouse-units we found in the exarysate (15 and 25 cc. respectively) a rather considerable amount of mouse-units, but less than (about one-fifth of) the calculated amount. But the dialysate too had become weaker than should be the case if the menformon had spread evenly over both exarysate and dialysate. The membrane must thus have retained part of the menformon. This hypothesis was confirmed by the result of efforts to extract it from the membranes again.

Chemical Properties.

The difficulty with which at the present moment larger amounts of the purest "menformon" are available makes it impossible to say much

about its composition. Results with impure preparations are only valuable if they are negative. If, for instance, the impure preparation does not contain phosphorus, the pure product will certainly have none, etc.

Dry residue.

Because, on evaporating unto dryness in the usual manner at 100—110°, amounts of, say 2 cc. of an aqueous solution, containing 20 or more mouse-units, often leave only unweighable quantities of solids, we thought of the possibility of sublimation, and for that reason we dried at temperatures not over 50°. But with these precautions too the residue of 10 cc. (= 160 mouse-units) was not more, and no sublimate was demonstrable with certainty on a watchglass kept above it.

On evaporating 5 ccm. (= 18 mouse-units) in an exsiccator at 37° and at 15°, there was left less than 0.1 mg., probably even less than 0.01 mg. of residue (roughly estimated on comparison with known amounts of other substances). The unweighable residue from other preparations, present, for instance, as rings on a watchglass, did not show any change on further drying at 98°.

In those cases where hardly any residue was obtained (e.g. from 10 cc., containing at least 33 mouse-units) dissolving it again in 10 cc. of water yielded an active (though perhaps somewhat weakened) preparation. From a chemical point of view we thought this preparation somewhat suspect, but it should be borne in mind that also 0.01 mg. of adrenaline and even less are active in adult men.

Slightly impurer preparations (0.06 mg. per mouse-unit), obtained in an exsiccator at 37° (2.2 mg. from 10 cc.) showed a few crystals. Up to this moment we have been unable to identify these. In view of our experience that much purer preparations exist, we think it improbable that *these* crystals have anything to do with menformon itself.

Reactions on proteins, NH₂- or OH-groups; N-, P-, S- and cholesterol-content ¹⁾.

Relatively impure preparations with 0.04 % of dry residue (7—10 M.U. per ccm.) did not show the least *protein*-reaction (Heller's ring-, sulfo-salicylic-, ninhydrin-, biuret-reaction). 2 cc. of a similar preparation yielded no *nitrogen* on determination with the micro-Kjeldahl-method; diazo-benzene sulfonic acid gave a negative reaction on phenol- or aromatic NH₂-groups.

An impure specimen with 3—4 M.U. per cc. and a dry residue of 0.076 % (i.e. about 0.22 mg. per M.U.) yielded 6.6 mg. from 30 cc. on drying at 50°. One-third of this quantity was analysed and showed a nitrogen-content of 2.4 %.

¹⁾ Experiments by Miss E. DINGEMANSE, Ch. D.

It was impossible to demonstrate the presence of *phosphorus* in 2.2 mg., whereas 3.5 mg. of casein with about 0.027 mg. of phosphorus yields a positive reaction.

The nitrogen-content could point to a lecithin-like constitution, but the absence of phosphorus excludes this. Probably the nitrogen too is due to the presence of an impurity.

About the *sulphur*-content we cannot yet say anything: the impure product does not contain demonstrable amounts, but in the same quantity of protein it is quite as impossible to show its presence.

Cholesterol: 2.2 mg. of the impure product, dissolved in chloroform, give a negative cholesterol reaction on addition of acetic anhydride and sulfuric acid, whereas 0.1 mg. of cholesterol gives an unmistakable red or violet colouration. SALKOWSKI's reaction yielded the same negative result. The cholesterol content of the impure substance, if at all present, is thus certainly below 5 %. Addition of 0.3 % of *trikresol*, or of a 0.9 % solution of *sodium chloride* do not alter the aspect of a solution of menformon, nor its potency.

Keeping qualities: After being kept for 3 weeks in an incubator at 37° the solution did not show any change of potency.

We are of opinion, that in menformon we found a new substance, of which we hope to be able to publish something more positive in the near future. This makes it necessary for us to have larger quantities at our disposal which is a matter of considerable trouble considering the difficulty in obtaining the original material.

Physiological action and pharmacological assay.

Repeated injections into *mice* of one or more mouse-units (up to this moment we gave about 4—8 M.U. every day) seem to have no action, apart from the specific effect.

In *rabbits* subcutaneous or intravenous injection of 8—80 M.U. (= 1 to 10 cc), at one time or within 10 minutes, is supported without any ill effects, even when repeated within a few days.

In *men* injection of 1 cc. (= 3—45 M.U.) even when repeated, does not produce any general effects, as experiments on ourselves, as well as those taken by clinical men among our friends on female persons have shown. For this purpose we used solutions to which 0.3 % *trikresol* and 0.9 % NaCl had been added, and of which the sterility had been proved.

Blood-pressure and Heart.

Impure preparations (such as contained more solids than is tolerated by our definition of menformon), when injected intravenously into rabbits in urethane anesthesia, produced a steep lowering of their blood-pressure: on repeating the injection they died. The curves recently published by DICKENS and DODDS c.s. showing repeated lowering of blood-pressure

after intravenous injection are obtained with preparations of considerable impurity as compared with menformon. Only somewhat purer preparations, containing about 5 M.U. per mg. and therefore still too impure as to be comparable with our definition of menformon were *completely devoid of any action* upon the blood-pressure of rabbits or of cats decerebrated with novocain, even after injecting three times 1 cc., i.e. all together 6.6 mg. of solids and about 15 M.U.

Three and more M.U. did not show any definite effect on the isolated frog heart, beating at the Straub cannula.

Blood vessels.

On perfusing a LAEWEN—TRENDELENBURG—KOCHMANN preparation (frog or guinea-pig) we were unable to find definite changes of the width of the blood vessels.

Respiration.

An impure preparation showed some effect upon the respiration of rabbits in urethane anesthesia, along with a lowering of their blood-pressure. Pure preparations, however, do not produce the least alteration of the respiration.

Uterus.

Doses of about 17 M.U. often produce an unmistakable *contraction* in the *isolated uterus* of virgin guinea pigs. The amount of solids in this dose was far below 0.1 mg.; smaller doses did not act with certainty.

The *uterus in situ*, as studied by means of the abdominal window (through which several foeti could be clearly seen), did not show any change on intravenous injection of about 45 M.U. Repetition of the experiment 3 days afterwards remained negative too. The young were born the following day, i.e. about 2—3 days before the normal end of pregnancy. Of course this does not necessarily lead to the conclusion that menformon was the cause, for a similar early birth often occurs without any injections.

Blood sugar.

Injection of 15 M.U. was *without any influence* upon the blood sugar content of rabbits.

Growth.

In spring we performed some experiments with unchanged follicular liquid upon tadpoles. An addition of 10 % of the liquid to the growing medium killed the animals whereas a content of 2 % was without any

influence. On the contrary an amount of only 8 M.U. in total is capable of causing within 10 days an enlargement of more than 200 % of the generative organs in young female rats scarcely capable of holding their own in absence of their mother, as compared with the control animals.

Quite similar results were obtained with young guinea pigs. The control animals were injected with a liver extract prepared in quite the same way and in doses quite equal as regards the quantity of raw material (follicular liquid and liver) to which they corresponded. The animals got 7 injections of 0.2 cc. each within a week. The control animal got altogether 0.06 mg. of solids, the others about 23 M.U. with unweighable residue. In this case too the generative organs of the menformon-animals had a weight which by more than 100 % surpassed that of the controls. This holds good for the animals killed immediately after the last injection, as well as for those which were killed a few days afterwards. This last fact, we think, pleads more in favour of a stimulation of the growth of the generative organs than for the induction of temporary estrous changes.

Summary.

A simple method is described (the "water-method") to prepare a substance (from follicular fluid especially) which is able to induce in castrated mice unmistakable changes of the vagina and uterus that are probably identical with those of the spontaneous estrous cycle. This substance is defined as one in which 1 mgr. of solids represent at least 10 of the so called mouse-units = M.U. These mouse-units are defined, and the conditions for standardization are given.

Provisionally chemical analysis shows the substance to be protein-free, probably also free from nitrogen, from cholesterol, from phosphorus and possibly also from sulphur. Besides in volatile solvents it is readily soluble in water. The name *menformon* is proposed.

Pharmacologically menformon in doses up to 45 M.U. showed perfectly non-toxic in men and animals; intravenous injections up to 80 M.U. did not affect blood pressure nor respiration in any way. Doses of 17 M.U. often caused contractions in the isolated virgin uterus. Up to this moment our observations on the uterus in situ (rabbits with abdominal window) did not show any influence. The effect upon the growth of the generative organs in young, normal female rats and guinea-pigs is evident.

The wide-spread belief that the substances which cause changes similar to the spontaneous estrous cycle are not in true solution in water and the observation that they act upon the blood-pressure both probably took their origin in a failure to free them from impurities, from lipoids especially.

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November 1925.

Physics. — "*The analogue of Clapeyron's law in the case of evaporating electrons*". By J. DROSTE. (Communicated by Prof. H. A. LORENTZ).

(Communicated at the meeting of October 31, 1925).

This paper contains the approved solution of a question, proposed in 1924 by the Mathematical Society (Wiskundig Genootschap) at Amsterdam as to the validity of an equation, mentioned further on as equation (7) and relating to the equilibrium of a hot metal with the electrons it emits. This equation, first derived by RICHARDSON¹⁾ on a statistical basis, is easily verified in those cases in which the electric potential can be calculated. It was asked whether or not this equation holds good in the case of an arbitrarily shaped evacuated enclosure bounded by bodies of one and the same metal. It will be found that the question has to be answered in the affirmative.

1. We consider a vacuum bounded by bodies of a given hot metal emitting electrons. In the equilibrium state there will be a temperature T , constant throughout the system. By means of wires of the same metal the electric potential of all the bodies is made equal. As these wires make of all the bodies a single one, we may speak simply of the conductor and its surface; it is supposed only that the wires do not resist relative displacements of the bodies.

In the equilibrium state the electronic gas will have a concentration N , which at the surface has the value N_0 , a quantity depending only on T . The pressure in the gas will be $p = kNT$. As the lines of force end normally at the surface the stress along them will give rise to a normal force acting on the surface, amounting to $\frac{1}{2} \left(\frac{\partial \varphi}{\partial n} \right)^2$ per unit area, φ representing the electric potential. So the total pressure will be normal and have the value

$$A = kN_0T - \frac{1}{2} \left(\frac{\partial \varphi}{\partial n} \right)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

2. Now consider an infinitesimal, infinitesimally slow change in the system with the temperature T changing by δT and the shape of the space R changing by infinitesimal displacements of the bodies.

This may be imagined as due to an infinitesimal increase $\delta \lambda$ in a parameter λ which, with other parameters, determines the relative position of the bodies. An arbitrary point of the surface will consequently get

¹⁾ Vid. O. W. RICHARDSON, The electron theory of matter (1914), pag. 447, eq. (14),

an infinitesimal displacement, the projection of which on the normal at the surface we call δa ; we suppose this normal to be directed from the space R toward the interior of the conductor.

The work done by the system in consequence of these displacements amounts to

$$\int A \delta a d\sigma = \delta\lambda \int A \frac{\delta a}{\delta\lambda} d\sigma,$$

in which the integral has to be taken over the surface of the conductor.

The energy U of the system being a function of T and λ , it follows from the second law of Thermodynamics that

$$\frac{1}{T} \left(\delta U + \delta\lambda \int A \frac{\delta a}{\delta\lambda} d\sigma \right)$$

or, what reduces to the same thing,

$$\frac{U}{T^2} \delta T + \left(\int \frac{A}{T} \frac{\delta a}{\delta\lambda} d\sigma \right) d\lambda,$$

derived from the former expression by subtracting $\delta \left(\frac{U}{T} \right)$, is an exact differential. This leads to the relation

$$\frac{\partial}{\partial\lambda} \left(\frac{U}{T^2} \right) = \frac{\partial}{\partial T} \int \frac{A}{T} \frac{\delta a}{\delta\lambda} d\sigma$$

or

$$\frac{\partial U}{\partial\lambda} = \int \left(T \frac{\partial A}{\partial T} - A \right) \frac{\delta a}{\delta\lambda} d\sigma.$$

We write this equation in the form

$$\delta_x U = \int \left(T \frac{\partial A}{\partial T} - A \right) \delta a d\sigma, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where δ_x means an increment corresponding to $\delta\lambda$ (or, what reduces to the same thing, to the displacements δa), T remaining constant. The same symbol will stand for the increment which a quantity depending on the coordinates gets in consequence of $\delta\lambda$, T as well as the coordinates remaining constant.

3. We now proceed to calculate both members of (2). If the evaporation of an electron produces a constant increase ε of energy, the left hand member of (2) will be

$$\delta_x U = \delta_x \int \frac{1}{2} (\text{grad } \varphi)^2 dS + \varepsilon \delta_x \int N dS;$$

both integrals are to be taken over the space R .

Now the displacements δa , taking place at constant temperature, will give rise to another distribution of the electrons and also to another value of φ ; let $N + \delta_x N$ be the new concentration, $\varphi + \delta_x \varphi$ the new

potential. As N_0 depends only on T , δN will be zero at the surface. The change of $\delta_a U$ consists of two parts:

$$\delta_a U = \int \delta_a \left\{ \frac{1}{2} (\text{grad } \varphi)^2 + \epsilon N \right\} dS + \int \left\{ \frac{1}{2} \left(\frac{\partial \varphi}{\partial n} \right)^2 + \epsilon N_0 \right\} \delta a d\sigma.$$

In the former part one has to do with the integral, taken over the original space R , of the increment which the quantity $\frac{1}{2} (\text{grad } \varphi)^2 + \epsilon N$ gets in consequence of the displacements δa ; the second integral has to be taken over the surface of R and represents the alteration which U gets by constant φ and N in consequence of the addition and subtraction of small parts to R .

Now substituting in the right hand member of (2) the value of A , taken from (1), we get

$$\int \left(T \frac{\partial A}{\partial T} - A \right) \delta a d\sigma = \int \left\{ kT^2 \frac{dN_0}{dT} - \frac{1}{2} T \frac{\partial}{\partial T} \left(\frac{\partial \varphi}{\partial n} \right)^2 + \frac{1}{2} \left(\frac{\partial \varphi}{\partial n} \right)^2 \right\} \delta a d\sigma.$$

Equation (2) thus becomes

$$\begin{aligned} \int \left(\delta_a \left\{ \frac{1}{2} (\text{grad } \varphi)^2 + \epsilon N \right\} dS + \int \epsilon N_0 \delta a d\sigma \right) &= \\ &= \int \left\{ kT^2 \frac{dN_0}{dT} - \frac{1}{2} T \frac{\partial}{\partial T} \left(\frac{\partial \varphi}{\partial n} \right)^2 \right\} \delta a d\sigma, \end{aligned}$$

which may be written in the form

$$\begin{aligned} \int \delta_a \left\{ \frac{1}{2} (\text{grad } \varphi)^2 dS + T \frac{\partial}{\partial T} \int \frac{1}{2} \left(\frac{\partial \varphi}{\partial n} \right)^2 \delta a d\sigma \right. &= \\ &= \left(kT^2 \frac{dN_0}{dT} - \epsilon N_0 \right) \int \delta a d\sigma - \epsilon \int \delta_a N dS. \end{aligned}$$

But the increase of the number of electrons in R , arising from the displacements δa at constant temperature, is

$$\delta_a \int N dS = \int \delta_a N dS + N_0 \int \delta a d\sigma$$

and consequently it is permissible to write $\delta_a \int N dS = \int \delta_a N dS$ for $N_0 \int \delta a d\sigma$ in the right hand member of the equation just found, so that this member becomes

$$kT^2 \frac{d \log N_0}{dT} \left(\delta_a \int N dS - \int \delta_a N dS \right) - \epsilon \delta_a \int N dS$$

and the equation, after transposition of a term from the right side to the left, becomes

$$\begin{aligned} \int \delta_a \left\{ \frac{1}{2} (\text{grad } \varphi)^2 dS + kT^2 \frac{d \log N_0}{dT} \int \delta_a N dS + T \frac{\partial}{\partial T} \int \frac{1}{2} \left(\frac{\partial \varphi}{\partial n} \right)^2 \delta a d\sigma \right\} &= \\ &= \left(kT^2 \frac{d \log N_0}{dT} - \epsilon \right) \delta_a \int N dS \end{aligned} \quad (3)$$

3. It now will be proved that the left hand member of this equation is zero. To this purpose we begin with the equations

$$\Delta \varphi = N e, \quad (4)$$

$$N e \text{ grad } \varphi = \text{grad } p; \quad (5)$$

from these equations the first enables us to calculate φ if N be given, the second expresses the equilibrium condition for the electrons. By replacing p by $k N T$ equation (5) becomes

$$N e \text{ grad } \varphi = k T \text{ grad } N.$$

Now in φ an additive quantity independent of the coordinates does not matter and so we may take

$$\varphi = \frac{k T}{e} \log N \quad (6)$$

so that (4) becomes

$$\Delta \varphi = e e^{k T \varphi}, \quad (4a)$$

in which the two meanings of e are not to be confused.

The quantity φ is a function of the coordinates as well as of T and the parameter λ .

Differentiating (4a) with respect to T and λ successively we find

$$\Delta \frac{\partial \varphi}{\partial T} = \left(\frac{\partial \varphi}{\partial T} - \frac{\varphi}{T} \right) \frac{e^2}{k T} e^{k T \varphi}, \quad \Delta \frac{\partial \varphi}{\partial \lambda} = \frac{e^2}{k T} e^{k T \varphi} \frac{\partial \varphi}{\partial \lambda}.$$

Multiplying the first of these equations by $\frac{\partial \varphi}{\partial \lambda}$, the second by $\frac{\partial \varphi}{\partial T} - \frac{\varphi}{T}$ and subtracting the results we find

$$\frac{\partial \varphi}{\partial \lambda} \Delta \frac{\partial \varphi}{\partial T} = \left(\frac{\partial \varphi}{\partial T} - \frac{\varphi}{T} \right) \Delta \frac{\partial \varphi}{\partial \lambda}.$$

This may be written in the form

$$\begin{aligned} \text{div} \left(\frac{\partial \varphi}{\partial \lambda} \text{grad } \frac{\partial \varphi}{\partial T} \right) - \left(\text{grad } \frac{\partial \varphi}{\partial \lambda} \cdot \text{grad } \frac{\partial \varphi}{\partial T} \right) \\ = \text{div} \left\{ \left(\frac{\partial \varphi}{\partial T} - \frac{\varphi}{T} \right) \text{grad } \frac{\partial \varphi}{\partial \lambda} \right\} - \left(\text{grad } \left\{ \frac{\partial \varphi}{\partial T} - \frac{\varphi}{T} \right\} \cdot \text{grad } \frac{\partial \varphi}{\partial \lambda} \right) \end{aligned}$$

or

$$\frac{1}{T} \left(\text{grad } \varphi \cdot \text{grad } \frac{\partial \varphi}{\partial \lambda} \right) + \text{div} \left\{ \left(\frac{\partial \varphi}{\partial T} - \frac{\varphi}{T} \right) \text{grad } \frac{\partial \varphi}{\partial \lambda} \right\} - \text{div} \left(\frac{\partial \varphi}{\partial \lambda} \text{grad } \frac{\partial \varphi}{\partial T} \right) = 0.$$

Multiplying this equation by $T dS$ and integrating over R we get

$$\int \frac{\partial}{\partial \lambda} \frac{1}{2} (\text{grad } \varphi)^2 dS + \int \left(T \frac{\partial \varphi}{\partial T} - \varphi \right) \frac{\partial^2 \varphi}{\partial \lambda \partial n} d\sigma - \int T \frac{\partial \varphi}{\partial \lambda} \frac{\partial^2 \varphi}{\partial n \partial T} d\sigma = 0.$$

Now at the surface it follows from (6) that

$$T \frac{\partial \varphi}{\partial T} - \varphi = T \frac{\partial}{\partial T} \left(\frac{k T}{e} \log N \right) = \frac{k T}{e} \log N = \frac{k T^2}{e} \frac{\partial \log N}{\partial T} = \frac{k T^2}{e} \frac{d \log N_0}{dT}$$

and consequently the second term changes into

$$\frac{kT^2}{e} \frac{d \log N_0}{dT} \int \frac{\partial^2 \varphi}{\partial \lambda \partial n} d\sigma$$

or

$$\frac{kT^2}{e} \frac{d \log N_0}{dT} \int \Delta \frac{\partial \varphi}{\partial \lambda} dS.$$

Since $\Delta \frac{\partial \varphi}{\partial \lambda} = \frac{\partial}{\partial \lambda} \Delta \varphi = \frac{\partial}{\partial \lambda} (Ne)$ we may write

$$\int \frac{\partial}{\partial \lambda} \frac{1}{2} (\text{grad } \varphi)^2 dS + kT^2 \frac{d \log N_0}{dT} \int \frac{\partial N}{\partial \lambda} dS - T \int \frac{\partial^2 \varphi}{\partial n \partial T} \frac{\partial \varphi}{\partial \lambda} d\sigma = 0$$

and from this we find by multiplying by $\delta \lambda$

$$\int \delta_\lambda \frac{1}{2} (\text{grad } \varphi)^2 dS + kT^2 \frac{d \log N_0}{dT} \int \delta_\lambda N dS - T \int \frac{\partial^2 \varphi}{\partial n \partial T} \delta_\lambda \varphi d\sigma = 0.$$

Now at the surface

$$\delta_\lambda \varphi = - \frac{\partial \varphi}{\partial n} \delta a,$$

for we see from (6) that if T remains constant φ will not change at a point which moves with the surface. Substituting, the last term becomes

$$T \frac{\partial}{\partial T} \int \frac{1}{2} \left(\frac{\partial \varphi}{\partial n} \right)^2 \delta a d\sigma,$$

and we get an equation which says that the left hand member of (3) is zero. Consequently (3) becomes

$$\left(kT^2 \frac{d \log N_0}{dT} - \varepsilon \right) \delta_\lambda \int N dS = 0.$$

As it is always possible to displace the bodies in such a way that $\delta_\lambda \int N dS$ is not zero, i.e. that in consequence of the displacements electrons do evaporate or condense, we must have

$$kT^2 \frac{d \log N_0}{dT} - \varepsilon = 0$$

or

$$\frac{d \log N_0}{dT} = \frac{\varepsilon}{kT^2} \quad \dots \dots \dots (7)$$

Geophysics. — “*On the relation between wind or current and mean sealevel in the Indian and the Atlantic oceans and the adjacent seas.*” By P. H. GALLÉ. (Communicated by Dr. J. P. VAN DER STOK.)

(Communicated at the meeting of October 31, 1925).

1. *Introduction.*

In a former communication ¹⁾ in the Proceedings of this Academy the relation between the Trades of the North Atlantic and the mean sealevel in the seas of northern Europe has been investigated. Data about mean sealevel increased considerably since that time, this is the reason why this investigation has been taken up again.

2. *Bay of Bengal and Gulf of Aden.*

The yearly range of the mean level is extremely large at Kidderpore, situated 85 nautical miles up the river from the mouth of the Hooghly, which flows into the extreme north of the Bay of Bengal. Monthly departures from normal are :

—65, —72, —59, —43, —28, —5, +53, +106, +114, +57, —12, —46 ;

here and further on all departures are given in c.m..

Atmospheric pressure has some influence on the waterlevel, pressure in Calcutta shows the following yearly range in m.m. :

+5.9, +4.4, +1.8, —0.9, —3.3, —6.0, —6.3, —4.9, —2.4, +1.4, +4.6, +6.1,

and if mean sealevel at the head of the Bay was only dependent upon pressure, its range would be approximately :

—8, —6, —2, +1, +4, +8, +8, +6, +3, —2, —6, —8.

When we accept this range also for Kidderpore, it is evident that besides atmospheric pressure other influences are responsible for the following range in mean riverlevel at that place :

—57, —66, —57, —44, —32, —13, +45, +100, +111, +59, —6, —38.

The two principal phenomena responsible for this yearly range are :

¹⁾ These Proceedings 17, p. 1147.

10. The monsoons in the Bay of Bengal.

20. The enormous rainfall in this part of India.

At Calcutta and Cherrapunji the monthly rainfall in m.m. is consecutively:

7, 26, 29, 39, 142, 280, 313, 322, 264, 98, 16, 8;
19, 55, 280, 819, 1309, 2670, 2780, 1943, 1352, 355, 38, 6.

It is obvious that the height of the level in the Hooghly must rise or fall with the quantity of rain falling in the basin of the river, but the amount of this rise and fall is unknown to us. We tried to find a correction for the precipitation by calculating month for month the ratio between departure of level for primo May and departure of windforce for medio March, etc.; the windforce calculated in the way indicated in the Table on page 907.

The reason why a phasedifference of six weeks was applied will be found further on.

We intended to smooth the discrepancies between the monthly ratios by taking into account the rainfall, but the mutual differences between these ratios were rather large and did not decrease after a correction for the rainfall.

The amount of rainfall in this region however depends in no small degree upon the monsoon; rainfall and monsoon are not inter-independent; if the Southwest-monsoon is not steady or when its force is below the average the transport of moisted air from the Bay to the windside of the Himalaya-mountains is smaller than normal and rainfall shows a deficit.

The correlation between the energy of the wind in the Bay of Bengal and rainfall in Calcutta and Cherrapunji is consecutively:

$$\begin{array}{ll} r = +0.934 & f = \pm 0.024 \\ = +0.944 & = \pm 0.021 \end{array}$$

Wind is the prime factor and when we consider in our further calculations only the energy of the wind, precipitation is automatically taken into account.

Secondly we pointed to the monsoons as generators of fluctuations in the level.

From November to March the Northeastmonsoon, from May to September the Southwestmonsoon is blowing over the Bay, April and October are transitionmonths.

In connection with the trend of the coastline at the head of the Bay and near the mouth of the Hooghly, wind from the North is off-shore, a wind from the South an on-shore wind.

Monthly direction and force of the wind was computed for the area 10° — 20° N./ 80° — 90° E., the projections of these resultants on the direction North-South give what we call the acting or working components and their squares the monthly energy. An on-shore component is positive, an off-shore negative.

In the following table the different data are given, the force of the wind is given here and further on in Beaufort-units.

Month	Wind		Working component	Energy	Departures from normal	
	Direction	Force			Energy	Level
January	N 42°0	2.7	—2.01	—4.04	—5.27	— 57
February	39	1.3	—1.01	—1.02	—2.25	— 66
March	20	0.3	—0.28	—0.08	—1.31	— 57
April	293	1.2	—0.47	—0.22	—1.45	— 44
May	240	2.5	+1.25	+1.56	+0.33	— 32
June	233	4.3	+2.59	+6.71	+5.48	— 13
July	231	4.2	+2.64	+6.97	+5.74	+ 45
August	234	3.9	+2.29	+5.24	+4.01	+100
Sept.	233	3.0	+1.81	+3.24	+2.01	+111
October	173	0.9	+0.89	+0.79	—0.44	+ 59
Nov.	57	1.5	—0.82	—0.67	—1.90	— 6
Dec.	50	3.0	—1.93	—3.72	—4.95	— 38

It is a known fact that between monsoons and the driftcurrents they cause, a phasedifference exists of about one month.

The current in the Bay of Bengal being seasonally on- and off-shore and being responsible — at least for the greater part — for the range in meanlevel at Kidderpore, it is evident that this phasedifference must be taken into account.

For this reason the correlationfactor was calculated not only between the simultaneous departures from normal of the energy of the wind and the riverlevel, but also between the first mentioned departures in January, February, etc. and departures in the level in February, March; March, April etc.

The result is given here :

Simultaneously	$r = +0.618$	$f = \pm 0.121$
Phasediff. one month	$= +0.914$	$= \pm 0.031$
„ two „	$= +0.920$	$= \pm 0.031$
„ three „	$= +0.616$	$= \pm 0.121$

This seems to prove that fluctuations and changes in the windsystem in

the northern part of the Bay are followed after about six weeks by fluctuations in riverlevel at Kidderpore.

In the Gulf of Aden the yearly range of mean sealevel is fairly large, before and after correction for atmospheric pressure the following data show the monthly range of mean sealevel for rainless Aden :

+	5.6,	+	7.0,	+	8.1,	+	9.9,	+	10.4,	+	3.1,
—	6.4,	—	13.8,	—	12.6,	—	9.6,	—	3.9,	+	2.2;
+	10.7,	+	11.2,	+	10.2,	+	10.3,	+	9.0,	—	2.5,
—	13.2,	—	20.0,	—	15.7,	—	7.9,	+	0.1,	+	7.5,

and this fluctuation is in our opinion caused by the monsoons in the Arabian Sea at the mouth of the Gulf.

The wind was computed for the area 10° — 20° N./ 45° — 60° E. ; the monthly vectors were projected on the direction N. 70° E. for the calculation of the working components etc.

The result is given in the next table, a working component in westerly direction is positive, in easterly direction negative :

Month	Wind		Working component	Energy	Departures from normal	
	Direction	Force			Energy	Level
January	N. $67^{\circ}0$	2.8	+ 2.76	+ 7.62	+ 6.91	+ 10.7
February	75	2.6	+ 2.60	+ 6.76	+ 6.01	+ 11.2
March	84	2.2	+ 2.18	+ 4.75	+ 4.04	+ 10.2
April	78	1.3	+ 1.30	+ 1.69	+ 0.98	+ 10.3
May	182	0.9	— 0.23	— 0.05	— 0.76	+ 9.0
June	221	3.2	— 2.59	— 6.71	— 7.42	— 2.5
July	221	4.0	— 3.23	— 10.41	— 11.42	— 13.2
August	220	3.4	— 2.71	— 7.34	— 8.05	— 20.0
Septemb.	203	2.0	— 1.17	— 1.37	— 2.08	— 15.7
October	80	0.9	+ 0.89	+ 0.79	+ 0.08	— 7.9
Novemb.	59	2.3	+ 2.19	+ 4.80	+ 4.09	+ 0.1
Decemb.	64	2.9	+ 2.82	+ 7.95	+ 7.24	+ 7.5

The reason why in the case of Kidderpore the wind of January was considered in connection with the level of February, etc. is also valid for Aden.

Simultaneously	$r = +0.760$	$f = \pm 0.083$
Phasediff. one month	$= +0.957$	$= \pm 0.017$
„ two „	$= +0.875$	$= \pm 0.058$
„ three „	$= +0.543$	$= \pm 0.134$

Fluctuations and changes in the windsystem of the Arabian Sea are followed by changes in mean sealevel at Aden after about five weeks.

A very fair agreement exists between the results for Kidderpore with its abundant rainfall, and rainless Aden.

The level-data for Kidderpore result from the period 1881—1893, those for Aden from 1879—1893, a lapse of time long enough for the tropics ; the wind-data have been computed over the period 1854—1910.

Although a large correlationfactor alone does not prove the existence of a causal relation between the considered phenomena, in this case wind and mean sealevel, daily experience that on-shore and off-shore winds raise and lower the waterlevel forces us to make the conclusion that in the case of Kidderpore and Aden a causal relation exists.

North Atlantic Ocean.

In the Atlantic conditions are not so simple as in the Indian Ocean, they are on the contrary rather complicated.

The great variations in general weatherconditions over Western-Europe and over the northern part of the Atlantic depend for the greater part upon the permanent sub-tropical anticyclone, the tropical belt of low pressure and the semi-permanent Iceland-depression.

Variations in atmospheric pressure in the above-mentioned areas are followed by increased or decreased velocity of the Equatorial current, of the North Atlantic Stream and of its ramifications that flow into the North Sea, the Baltic and the Polar Basin. BREITFUSZ was the first to point to the fact that a period exists in the quantity of Atlantic water flowing into the Polar Basin, with a maximum in winter and spring, a minimum in summer and autumn.

BREITFUSZ's opinion is: „that this flow and ebb is not in the first place dependent upon local circumstances, but that they originate in the Gulfstream and in the many circumstances affecting this stream on its way, thousands of miles long, from its cradle under the equator through the Caribbean Sea, the Atlantic to the Polar Basin”.

Its „cradle under the equator” is the North-Equatorial current, a driftcurrent caused by the Northeast-trade. A Northeast-trade stronger than normal points to a sub-tropical anticyclone better developed than under average conditions and, as shown by HANN, a surplus of pressure in this region generally corresponds with a negative departure of pressure in the Iceland-region.

But under these conditions the Southwesterly air-transport between 40° and 50° northern latitude is stronger than normal and raises the speed and stability of the North Atlantic Stream.

Hitherto, in connection with European waterlevel, only more distant factors were discussed, but as well as for Kidderpore and Aden more local influences near the European coast exist, more or less responsible for fluctuations in mean sealevel.

However, these influences are originally dependent upon the above-named centres of action, and finally we come to the conclusion: Northeast-trade or Equatorial current prime *cause*, fluctuations in mean sealevel around the whole ocean *result*, but not directly connected.

It was tacitly accepted that for our purpose we might consider the driftcurrent instead of the wind, raising this current.

The correlationfactor between wind and current in 15°—25° N./25°—40° W. is $r = +0.69$ $f = \pm 0.13$; and if in the case of Aden the current in the Arabian Sea was taken into account, the following correlationfactors would have been the result:

Simultaneously	$r = + 0.629$	$f = \pm 0.118$
Phasediff. one month	$= + 0.906$	$= \pm 0.035$
„ two	$= + 0.880$	$= \pm 0.044$
„ three	$= + 0.606$	$= \pm 0.123$

Fluctuations in current and mean sealevel follow each other after a fortnight.

Instead of the wind, the current may be taken into account and for our further investigation in the Atlantic Ocean this will be done.

Departures from mean sealevel along the coasts of the Atlantic and adjacent seas are given in Table I; Table II contains the cosine-formulae, the epoch is January 15th, and generally the amplitude of the yearly range is larger than that of the semi-yearly.

The phase of the yearly range increases from 191° at Washington and 204° at Baltimore—Portland, to 245° at Horta, 300° at the entrance of the Channel, over 300° in the Northern part of the North Sea, to 320° at Kabelvaag and Vardö.

In the North Sea and Baltic we meet with the reverse:

The phase decreases from 281° at Katwijk, to 272° at Esbjerg, 262° in the Skager-rak, 229° from Kiel—Swinemünde and increases again along the Swedish and German coast, Stolpmünde—Pillau from 263° to 272°, to 292° at Utö and Lypertö along the Finnish coast.

The discontinuity in the regular course of this phase is probably caused by the big quantities of fresh water, originating from melting ice and snow in spring and early summer and from rain in summer and autumn, that flow into the Baltic, a typical mediterranean sea.

TABLE I.
YEARLY RANGE OF MEAN SEALEVEL IN CM.

Station	Period	Jan.	Feb.	Mrh.	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
Washington Old Point Comfort	— —	— 7.6	—10.9	—10.1	— 3.3	5.7	11.1	10.6	6.2	2.1	0.3	— 0.5	— 3.1
Baltimore . .	1903—1923	—12	—13	— 8	— 1	4	9	7	10	10	6	— 3	— 9
Atlantic City	1912—1920	— 7	— 9	— 8	— 1	2	4	3	6	6	7	1	— 3
St. Hamilton	1893—1920	— 8	—11	— 6	1	2	5	5	6	6	6	0	— 6
Portland . .	1912—1922	— 3	— 4	— 5	— 1	0	2	2	2	2	2	2	— 1
Sta. . . .	1905—1923	0.0	— 0.7	— 2.6	— 2.4	— 1.9	0.3	1.3	2.5	1.8	1.4	0.4	0.5
St. . . .	— —	2.9	— 2.5	— 6.1	— 6.0	— 4.4	— 2.5	— 2.1	— 1.1	1.1	5.2	8.0	7.5
Wyn. . .	1915—1925	1.1	0.7	— 0.7	— 4.2	— 3.2	— 5.8	— 2.7	— 0.9	0.9	5.2	5.7	3.8
Wijk . . } Lingen . }	1884—1920	1.7	— 3.6	— 6.1	— 8.3	— 9.1	— 4.8	0.4	4.4	3.3	7.2	5.8	8.9
Extowe . .	1917—1925	0.6	— 4.0	— 4.0	— 3.4	— 7.0	— 1.2	0.2	2.3	4.2	6.2	2.9	3.8
Enbar. . .	1913—1925	4.5	0.7	— 5.1	— 8.6	— 7.4	— 5.3	— 1.5	0.3	2.3	5.3	7.1	7.8
Andee . . .	1867—1912	6.7	0.9	— 5.4	— 9.1	—10.9	— 8.5	— 6.4	— 0.9	— 0.6	7.6	10.6	13.7
Ardeen . .	1862—1913	5.2	0.6	— 3.7	— 7.9	— 8.8	— 6.7	— 3.4	— 0.6	1.2	6.7	6.7	11.0
Merhaven.	1898—1924	3.2	— 2.3	— 9.0	— 7.3	— 9.2	— 0.6	3.7	7.2	3.8	0.3	5.2	5.2
Bjerg . . .	1889—1902	3.0	— 7.2	— 4.3	—13.6	—12.2	— 8.5	0.7	6.2	8.6	12.8	2.9	12.0
Shals. . .	1892—1911	1.6	— 0.7	—10.4	—10.0	—10.8	— 4.1	4.9	8.4	4.0	6.1	5.1	5.8
Derikshavn	1893—1911	2.7	— 1.6	— 9.4	— 9.6	—10.2	— 3.7	4.3	7.2	4.2	4.9	6.4	5.6
huus . . .	1888—1911	1.9	— 2.4	— 6.7	— 7.8	— 8.0	— 3.6	2.5	5.0	4.3	5.6	5.3	3.8
ricia . . .	1889—1911	— 0.1	— 1.9	— 5.5	— 6.2	— 5.7	— 2.9	2.1	4.1	5.1	5.0	3.3	2.7
Enbaek . .	1898—1911	3.2	— 1.9	—11.1	— 9.3	— 8.9	— 2.4	7.0	9.8	5.8	2.4	3.8	1.6
Enhagen .	1889—1911	0.5	— 1.1	— 7.5	— 9.1	— 8.4	— 2.8	5.7	8.5	7.1	3.7	2.0	1.7
Sör . . .	1897—1911	— 0.7	— 0.1	— 6.1	— 7.0	— 5.4	— 3.0	3.8	6.5	6.2	2.8	1.6	2.2
shavn . .	1896—1911	0.4	— 0.6	— 6.2	— 6.7	— 6.3	— 3.1	4.1	6.3	5.5	2.5	3.0	1.0
lser . . .	1898—1911	— 4.3	2.0	— 4.1	— 7.2	— 3.9	— 2.2	5.0	7.8	8.4	1.1	— 3.6	1.0
Berg. . .	1887—1900	— 2.6	— 6.3	— 7.8	—11.8	— 8.4	— 3.7	5.5	8.2	8.6	9.3	4.0	5.1
nd. . . .	1887—1900	— 4.9	1.2	— 5.5	— 8.4	— 8.2	— 3.3	4.1	5.4	6.4	7.1	1.4	4.4
skrona . .	1887—1900	— 5.4	2.1	— 6.6	— 9.8	—11.0	— 4.0	4.5	6.5	7.2	9.5	1.6	5.4
dsort . .	1887—1900	— 2.1	1.3	— 8.5	—12.9	—13.0	— 6.0	5.2	8.5	8.8	8.9	2.6	7.4

TABLE I. (Continued).

YEARLY RANGE OF MEAN SEALEVEL IN C.M.

Station	Period	Jan.	Feb.	Mrh.	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
Grönskär . .	1888—1900	— 1.4	1.9	— 8.7	—13.5	—13.8	— 7.1	5.0	8.4	10.1	9.2	4.0	0.6
Björn . . .	1891—1900	— 2.0	1.8	— 6.5	—11.9	—13.0	— 9.0	4.0	5.0	10.8	6.9	3.9	10.0
Droghällan .	1898—1900	4.8	1.6	— 9.2	—14.3	—15.7	—10.4	2.0	2.6	7.5	7.8	8.0	15.0
Ratan . . .	1891—1900	— 1.7	— 0.5	— 7.3	—12.6	—14.8	—10.7	3.4	4.5	11.6	8.6	8.7	11.0
Kiel	1895—1909	— 3.1	— 0.1	— 3.6	— 3.4	— 3.5	— 0.2	4.7	4.3	5.7	1.0	— 3.1	0.6
Travemünde .	1882—1924	— 3.7	— 1.1	— 3.9	— 4.0	— 2.4	1.1	4.8	5.7	6.0	1.5	— 3.5	— 0.6
Mariënleuchte	1882—1924	— 3.2	— 1.1	— 5.1	— 5.3	— 4.3	0.5	5.6	7.2	5.7	2.3	— 2.8	0.6
Wismar . . .	1882—1924	— 4.1	— 2.2	— 3.2	— 4.7	— 3.2	2.4	6.9	6.5	6.2	1.1	— 4.0	— 1.0
Warnemünde	1882—1924	— 3.1	— 1.5	— 5.5	— 5.8	— 4.7	1.2	6.9	7.4	6.7	1.8	— 2.7	— 0.6
Arkona . . .	1882—1924	— 0.5	0.2	— 5.7	— 8.0	— 7.9	— 2.5	4.1	7.3	6.4	3.2	— 0.2	3.0
Swinemünde .	1882—1924	— 2.5	— 0.1	— 6.2	— 6.6	— 6.2	0.0	6.6	8.8	6.6	0.1	— 3.5	3.0
Stolpmünde .	1911—1924	3.2	— 2.9	—10.2	— 9.4	—11.2	0.1	5.8	9.6	8.8	1.6	— 1.1	5.0
Pillau . . .	1898—1924	3.3	0.8	— 9.9	—11.4	—10.9	— 2.5	7.1	12.2	8.8	— 0.8	— 1.0	4.0
Memel . . .	1898—1918	4.8	2.1	— 9.6	— 4.8	—13.2	— 7.5	2.2	9.4	8.1	— 1.9	0.5	9.0
Christiania .	1886—1890 1904—1918	4.8	— 3.5	—11.4	— 9.9	— 8.1	— 1.2	5.3	9.0	3.2	2.5	4.8	4.0
Oscarsburg .	1872—1881	— 6.0	—16.9	—15.0	—13.6	— 8.4	— 5.6	12.0	11.0	17.6	10.9	7.4	6.0
Arendal . . .	1886—1889	— 0.2	—10.9	— 9.2	—11.3	— 6.3	— 1.5	4.7	7.6	5.6	6.7	6.9	7.0
Stavanger . .	1881—1885 1899—1905 1911—1918	3.3	0.9	— 6.1	— 8.5	— 7.1	— 7.9	— 5.0	2.7	1.8	6.8	10.3	8.0
Bergen . . .	1883—1889 1911—1918	7.8	— 0.1	— 7.7	—13.2	—11.7	— 8.4	— 3.9	1.3	4.0	6.5	12.9	12.0
Trondheim .	1872—1878 1880—1881	18.6	2.0	— 9.1	—16.4	—10.6	— 2.9	— 5.4	—11.5	— 3.7	9.1	18.2	11.0
Kabelvaag . .	1880—1883	22.3	10.1	— 5.5	—15.0	—10.5	—10.5	— 3.2	— 0.2	— 7.5	— 0.5	10.8	8.0
Narvik . . .	1905—1915	11.5	7.6	— 5.3	— 8.1	—13.6	— 9.7	— 6.6	— 6.0	0.3	5.6	11.9	12.0
Vardö . . .	1880—1883	18.2	10.0	— 5.5	—17.8	—14.1	— 8.4	— 0.6	4.7	— 4.0	0.2	9.9	7.0

In chapter 4, North Sea and Baltic data for quantities of this fresh water and precipitation are given.

The course of the phase of the semi-yearly range is in the beginning rather uncertain, at the entrance of the Channel the phase is about 240° ,

TABLE II. Formulae Yearly Range. Epoch 15 January.

1	Washington, Old Point Comfort	$+ 9.3 \cos (nt-191^\circ) + 4.1 \cos (2nt-161^\circ)$
2	Baltimore, Atlantic City, Fort Hamilton and Portland. . . .	$+ 7.4 \cos (nt-204^\circ) + 2.0 \cos (2nt-208^\circ)$
3	Horta	$+ 2.3 \cos (nt-245^\circ) + 0.9 \cos (2nt- 7^\circ)$
4	Brest.	$+ 6.2 \cos (nt-294^\circ) + 2.6 \cos (2nt-284^\circ)$
5	Newlyn	$+ 4.8 \cos (nt-307^\circ) + 0.7 \cos (2nt-191^\circ)$
6	Katwijk and Harlingen.	$+ 8.0 \cos (nt-281^\circ) + 1.6 \cos (2nt-345^\circ)$
7	Dumbar, Dundee, Aberdeen. . .	$+ 8.9 \cos (nt-306^\circ) + 1.7 \cos (2nt-327^\circ)$
8	Bremerhaven, Esbjerg, Hirshals .	$+ 8.8 \cos (nt-272^\circ) + 3.0 \cos (2nt- 16^\circ)$
9	Hirshals, Fredericshavn, Varberg, Arendal, Stavanger, Bergen. .	$+ 9.1 \cos (nt-262^\circ) + 1.9 \cos (2nt- 3^\circ)$
10	Aarhus, Fredericia, Hornback, Kopenhagen, Korsör, Slipshavn	$+ 6.8 \cos (nt-261^\circ) + 1.7 \cos (2nt- 33^\circ)$
11	Gjedser, Ystad, Karlskrona . . .	$+ 6.4 \cos (nt-258^\circ) + 2.4 \cos (2nt- 60^\circ)$
12	Kiel, Travemünde, Mariënleuchte, Wismar, Warnemünde, Arkona, Swinemünde	$+ 4.9 \cos (nt-229^\circ) + 2.9 \cos (2nt- 42^\circ)$
13	Stolpmünde, Pillau, Memel . . .	$+ 7.4 \cos (nt-263^\circ) + 5.8 \cos (2nt- 29^\circ)$
14	Varberg, Ystad, Karlskrona, Land- sort, Grönskär, Björn, Drög- hällan, Ratan	$+ 9.8 \cos (nt-272^\circ) + 3.0 \cos (2nt- 44^\circ)$
15	Utö en Lypertö. 1892—1900 . .	$+ 11.2 \cos (nt-292^\circ) + 3.5 \cos (2nt- 75^\circ)$
16	Stavanger, Bergen, Trondheim. .	$+ 10.8 \cos (nt-304^\circ) + 3.1 \cos (2nt-323^\circ)$
17	Bodö, Kabelvaag, Narvik, Vardö	$+ 12.3 \cos (nt-322^\circ) + 4.7 \cos (2nt-355^\circ)$

along the North Sea coasts and the Norwegian coast between 300° and 382° , in the Kattegat, the Sount and the Belts it fluctuates between 3° and 60° .

For the observations from Horta, Felixtowe, Dumbar and Newlyn, the German and Norwegian Coast we are indebted to the Director of the Meteorological Office at Ponta Delgada, the Officer in Charge of the Levelling Department of the Ordnance Survey Office at Southampton, the Director of the Geodetical Institute at Potsdam and the Director of Norges Geografiske Opmaaling at Kristiania.

The Dutch observations were provided by Algemeenen Dienst van den Rijkswaterstaat, the Hague, for the origin of other observations we point to the appendix "Literature".

In the region 15° — 25° N./ 25° — 40° W., part of "the cradle of the Gulfstream", monthly departures from normal, in miles per 24 hours, in the velocity of the North-Equatorial current run as follows:

-1.3, -1.8, -1.2, -0.9, -0.3, +0.8,
 +2.7, +1.3, +1.6, +0.5, -0.1, -1.3;

they are computed over the period 1855—1905.

For various stations the following correlation factors were calculated:

Washington, Old Point	Simultaneously	$r = +0.829$	$f = \pm 0.062^1)$
Comfort	Ph. Diff. one month	$= +0.551$	$= \pm 0.138$
Baltimore, Atlantic City,	Simult.	$= +0.869$	$= \pm 0.044$
Fort Hamilton, Portland	One month	$= +0.776$	$= \pm 0.076$
Horta	Simult.	$= +0.668$	$= \pm 0.107$
	One month	$= +0.889$	$= \pm 0.041$
	Two "	$= +0.778$	$= \pm 0.076$
Newlyn (Lands' End)	Two "	$= +0.649$	$= \pm 0.114$
	Three "	$= +0.896$	$= \pm 0.038$
	Four "	$= +0.907$	$= \pm 0.035$
	Five "	$= +0.601$	$= \pm 0.124$
Brest.	Two "	$= +0.725$	$= \pm 0.093$
	Three "	$= +0.885$	$= \pm 0.041$
	Four "	$= +0.801$	$= \pm 0.069$
Aberdeen	Two "	$= +0.636$	$= \pm 0.114$
	Three "	$= +0.921$	$= \pm 0.027$
	Four "	$= +0.896$	$= \pm 0.048$
Dundee	Two "	$= +0.584$	$= \pm 0.128$
	Three "	$= +0.896$	$= \pm 0.038$
	Four "	$= +0.908$	$= \pm 0.035$
	Five "	$= +0.715$	$= \pm 0.093$
Stavanger	Two "	$= +0.745$	$= \pm 0.086$
	Three "	$= +0.907$	$= \pm 0.035$
	Four "	$= +0.847$	$= \pm 0.055$
Narvik	Three "	$= +0.809$	$= \pm 0.069$
	Four "	$= +0.935$	$= \pm 0.024$
	Five "	$= +0.805$	$= \pm 0.069$

The general result is a big correlation factor and a phasedifference increasing when considering stations at higher latitude and greater longitude.

Though we cannot imagine a direct relation in such a sense, that a fluctuation in the speed of the Equatorial current is followed after 3, 4 or

¹⁾ A systematical error in the calculations is the cause why the correlation factors between the monthly departures of the velocity of the Equatorial Current and those of the water-level are generally a trifle (0.02) too small in the Proceedings in Dutch.

5 months by analogous fluctuations in supply of Atlantic water along the European coast, simply by direct watertransport, a relation exists so far that the causes, modifying the Equatorial current at the south-side of the sub-tropical anticyclone, modify at the same time the southwesterly wind at its north-side as was explained before.

These southwesterly winds transport smaller or bigger quantities than normal of relatively warm water and air to regions situated more to the North and East. The relatively warm water will have a great influence on the general air-circulation in these northerly regions and this process will continue for a considerable time.

In last instance this general aircirculation affects substantially the watertransport along the European coast. We agree with BREITFUSZ that the supply of water from the ocean to the European coasts and into Polar Basin is of a periodical nature.

4. *North Sea and Baltic.*

Leaving out of consideration the northern part of the British North Sea coast, the main peculiarity of these seas is the welldefined secondary minimum in October and November.

This phenomenon is very clearly to be seen along the German and Swedish coast, not so clearly along the Skagerrak, Kattegat, Sount, Belts and the Dutch coast. In the first named region, the relation between the amplitudes of the yearly and semi-yearly range is as 2.2 to 1, in the last named as 4.5 to 1.

We think we may point to the large amount of fresh water from melting snow and ice in spring and from the large precipitation in summer, that flows into the Baltic, when looking for the cause of this strong semi-yearly range in this sea. WITTING gives the following data for the supply of fresh water (K.M.³) and precipitation (m.m.) for the Bothnian Gulf.

	J.	F.	M.	A.	M.	J.	J.	A.	S.	O.	N.	D.
Fresh water	7	5	6	19	49	30	20	24	18	15	10	9
Precipitation	26	22	20	37	37	48	55	94	42	50	34	31

In the North Sea and the Baltic we find also a big correlationfactor between Equatorial current and mean sealevel, but in these narrow seas local influences play a more important part especially in the Baltic, where conditions for water-circulation are rather unfavourable.

The monthly resultant wind on the Dutch coast was computed from observations made from 1882—1910 on board the lightvessels Terschel-

lingerbank and Schouwenbank, the resultants were projected on the direction N 276° E. the squares of these projections gave the energy; an easterly direction is positive.

The data are given in the following table with monthly departures from mean level over the same period for a combination of Katwijk and Harlingen.

Month	Wind		Working compt.	Energy	Departures from normal	
	Direction	Force			Energy	Level
January	N 217° E	1.01	0.65	0.42	+0.02	+1.9
February	224	0.70	0.52	0.27	-0.13	-4.0
March	242	0.50	0.45	0.20	-0.20	-6.5
April	329	0.44	0.21	0.04	-0.36	-8.9
May	324	0.64	0.35	0.12	-0.28	-7.6
June	309	0.70	0.52	0.27	-0.13	-5.4
July	268	0.96	0.96	0.90	+0.50	+0.8
August	249	1.21	1.15	1.32	+0.92	+4.9
September	258	0.65	0.63	0.40	+0.00	+3.2
October	222	0.91	0.64	0.41	+0.01	+9.2
November	206	0.70	0.34	0.12	-0.28	+4.6
December	208	1.19	0.61	0.37	-0.03	+7.4

The correlation factor is:

$$\begin{array}{lll} \text{Simultaneously} & r = +0.472 & f = +0.149 \\ \text{Phasediff. one month} & = +0.398 & = +0.163 \end{array}$$

From observations made at Swambister, Bergen, Skudesnaes, Flushing and Helder, a resultant windforce "the North Sea wind" was computed. Monthly departures from normal of the energy show the following range:

$$\begin{array}{cccccc} +0.72, & -0.38, & -0.26, & -2.29, & -0.37, & -0.45, \\ +1.45, & +1.41, & +0.94, & -0.19, & -0.81, & +0.31, \end{array}$$

and the correlationfactor between these departures and those for mean sealevel at Harlingen and Katwijk (1884—1920) is:

$$\begin{array}{lll} \text{Simultaneously} & r = +0.478 & f = \pm 0.149 \\ \text{Phasediff. one month} & = +0.442 & = \pm 0.156 \\ \text{Simultaneously} & r = +0.781 & f = \pm 0.076 \\ \text{Phasediff. one month} & = +0.578 & = \pm 0.128 \end{array}$$

and for the Swedish coast from Varberg till Ratan:

Simultaneously	$r = +0.624$	$f = \pm 0.117$
Phasediff. one month	$= +0.615$	$= \pm 0.121$

The correlationfactor is the biggest when simultaneous departures are considered; in the North Sea the relation is much smaller than in the Baltic. For the correlationfactor between departures in the velocity of the Equatorial current and those in mean sealevel in the North Sea and the Baltic, the result is:

Katwijk and Harlingen:

Phasediff. one month	$= +0.624$	$= \pm 0.114$
„ two „	$= +0.806$	$= \pm 0.069$
„ three „	$= +0.880$	$= \pm 0.045$
„ four „	$= +0.648$	$= \pm 0.131$

Travemünde-Swinemünde:

Simultaneously	$r = +0.771$	$f = \pm 0.080$
Phasediff. one month	$= +0.833$	$= \pm 0.059$
„ two „	$= +0.600$	$= \pm 0.125$

Varberg-Ratan:

Phasediff. one month	$= +0.740$	$= \pm 0.086$
„ two „	$= +0.841$	$= \pm 0.055$
„ three „	$= +0.798$	$= \pm 0.069$

5. Conclusions.

1. Range in mean sealevel in the Indian and Atlantic Ocean and the adjacent seas is for the greater part a result of fluctuations of wind and current in regions, either far away from, or more in the neighbourhood of the place where mean sealevel is considered.

2. The Northeast-Trade or the Equatorial current is not the prime cause of fluctuations in mean sealevel on the European coast, but the regions of high and low pressure, that cause this wind and current, are substantially the generators of these fluctuations.

3. The correlationfactor between monthly departures of the velocity of the current or energy of the wind at a great or fairly great distance and departures in mean sealevel is generally large and varies between $+0.829$ and $+0.957$.

4. The correlationfactor between monthly departures of the local

wind in the North Sea and departures in mean sealevel in the North Sea and the Baltic varies between $+0.472$ and $+0.781$.

5. The phasedifference between monthly departures of the velocity of the Equatorial current and those of mean sealevel in the Atlantic Ocean and European coasts increases gradually :

Northeast part American coast	Simultaneous
Horta	one month
Channel	three months
North Sea	three months
Northern part Norwegian Coast	four months.

A discontinuity in the regular course of this phasedifference exists in the Baltic.

German coast	one month
Swedish coast	two months.

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In the case of linear correlation the probability formula runs:

$$dW = \frac{h h' \sqrt{1-\gamma^2}}{\pi} e^{-(h^2 u^2 - 2\gamma h h' u u' + h'^2 u'^2)} du \cdot du' \quad . \quad . \quad (11)$$

being

$$u = x - \bar{\xi} \quad , \quad u' = x' - \bar{\xi}'.$$

We have now the empirical data

$$Y_{kl} = \int_{x_{k-1}}^{x_k} \int_{x'_{l-1}}^{x'_l} D \cdot dx \cdot dx' = \int_{u_{k-1}}^{u_k} \int_{u'_{l-1}}^{u'_l} \frac{h h' \sqrt{1-\gamma^2}}{\pi} e^{-(h^2 u^2 - 2\gamma h h' u u' + h'^2 u'^2)} du \cdot du'$$

being

$$u_k = x_k - \bar{\xi} \quad , \quad u'_l = x'_l - \bar{\xi}'$$

For the extreme limits we must take:

$$\begin{aligned} x_0 &= -\infty, & x_n &= +\infty, & x'_0 &= -\infty, & x'_n &= +\infty, \\ u_0 &= -\infty, & u_n &= +\infty, & u'_0 &= -\infty, & u'_n &= +\infty. \end{aligned}$$

By substituting

$$hu = t \quad , \quad h'u' = t'$$

we arrive direct at the unimodular probability formula:

$$y_{kl} = \int_{t_{k-1}}^{t_k} \int_{t'_{l-1}}^{t'_l} \frac{\sqrt{1-\gamma^2}}{\pi} e^{-(t^2 - 2\gamma t t' + t'^2)} dt \cdot dt'.$$

Putting

$$\sqrt{1-\gamma^2} \cdot t = z \quad , \quad t' - \gamma t = \zeta,$$

we find (cf. the canonic form I)

$$dW = \frac{1}{\sqrt{\pi}} e^{-z^2} dz \cdot \frac{1}{\sqrt{\pi}} e^{-\zeta^2} d\zeta = ds \cdot d\sigma,$$

being written

$$s = \Theta(z) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^z e^{-v^2} dv \quad , \quad \sigma = \Theta(\zeta) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\zeta} e^{-v^2} dv$$

In this case z and s are functions only of t , hence only of x ; on the contrary ζ and σ are functions both of t and t' , hence both of x and x' .

To $x_0 = -\infty$ corresponds $t_0 = -\infty$, $z_0 = -\infty$, $s_0 = 0$;

„ $x_n = +\infty$ „ $t_n = +\infty$, $z_n = +\infty$, $s_n = 1$.

To $x'_0 = -\infty$ and x finite corresponds $t'_0 = -\infty$ and t finite, thus $\zeta = -\infty$, $\sigma = 0$.

To $x'_{n'} = +\infty$ and x finite corresponds $t'_{n'} = +\infty$ and t finite, thus $\zeta = +\infty$, $\sigma = 1$.

Introducing the notation

$$\begin{aligned} z(x_k) &= z_k & \zeta(x_k, x'_{l'}) &= \zeta_{kl} \\ s(x_k) &= s_k & \sigma(x_k, x'_{l'}) &= \sigma_{kl} \end{aligned}$$

with

$$\begin{aligned} z_0 &= -\infty & z_n &= +\infty & \zeta_{k0} &= -\infty & \zeta_{kn'} &= +\infty \\ s_0 &= 0 & s_n &= 1 & \sigma_{k0} &= 0 & \sigma_{kn'} &= 1, \end{aligned}$$

we obtain

$$\int_{x=-\infty}^{x_k} \int_{x'=-\infty}^{x'_l} D \cdot dx \cdot dx' = \frac{1}{\sqrt{\pi}} \int_{z=-\infty}^{z_k} e^{-z^2} dz \cdot \frac{1}{\sqrt{\pi}} \int_{\zeta=-\infty}^{\zeta_{k,l}} d\zeta = s_k \cdot \sigma_{kl},$$

whence

$$\sum_{k=1}^k \sum_{j=1}^l \int_{x_{l-1}}^{x_l} \int_{x'_{j-1}}^{x'_j} D \cdot dx \cdot dx' = s_k \cdot \sigma_{kl}. \quad (12)$$

We can now get rid of t' (i.e. of x') by integrating over x' (i.e. over t') between the extreme limits $x'_0 = -\infty$ and $x'_{n'} = +\infty$ (resp. $t'_0 = -\infty$ and $t'_{n'} = +\infty$); so we find

$$\int_{x=-\infty}^{x_k} \int_{x'=-\infty}^{+\infty} D \cdot dx \cdot dx' = \sum_{i=1}^k \sum_{j=1}^{n'} \int_{x_{i-1}}^{x_i} \int_{x'_{j-1}}^{x'_j} D \cdot dx \cdot dx' = s_k \cdot \sigma_{kn'} = s_k \times 1 = s_k \quad (13)$$

This integral obviously is equal to the sum of the frequency contents of all the rows $x = \xi_1$, $x = \xi_2$, ..., $x = \xi_k$, divided by the total number N .

So the value of s_k results from

$$s_k = \frac{\sum_{i=1}^k \sum_{j=1}^{n'} Y_{ij}}{N} \quad (14)$$

On the other hand the integral $\int_{x=-\infty}^{x_k} \int_{x'=-\infty}^{+\infty} D \cdot dx \cdot dx'$ is a function of x_k .

We therefore find to each x_k a value of s_k , i. e. a value of the

function $s(x)$. Furthermore the relation $s = \Theta(z)$ conjugates to each s_k a value of z_k .

Besides the pairs $s_0 = 0, z_0 = -\infty$ and $s_n = 1, z_n = +\infty$, we thus obtained $n-1$ pairs

$$(s_1, z_1), (s_2, z_2) \dots (s_{n-1}, z_{n-1}).$$

If the correlation is really linear, all these $n-1$ values of z must satisfy the linear relation:

$$z = \sqrt{1-\gamma^2} \cdot t = \sqrt{1-\gamma^2} \cdot hu = \sqrt{1-\gamma^2} \cdot h(x-\xi) = ax + b.$$

By interpreting x and z as rectilinear coordinates, the points (x, z) must be found on the straight line $z = ax + b$.

Inversely the collinear disposition of the points (x, z) empirically determined points to the possibility of linear correlation between x and x' . Properly speaking this only shows that the quantity x , considered by itself, is distributed according to the normal law.

As the function $z(x)$ empirically found (which in this case is linear) is wholly independent of x' , it gives us no indication at all, neither with regard to x' , nor to the connection between x and x' .

To determine the relation between x and x' we must appeal to the function $\sigma(x, x')$.

Integrating $\iint D dx dx' = \iint ds \cdot d\sigma$ over an infinitely narrow strip along $x = x_i$, the breadth of which is Δx_i , the integral being taken over x' from $x'_0 = -\infty$ to x'_1 , we find the value of this integral to be

$$L_{i1} = \Delta x_i \cdot \int_{x'=-\infty}^{x'_1} D(x_i, x') dx' = \Delta x_i \cdot \int_{x'=-\infty}^{x'_1} \left\{ \frac{ds}{dx} \cdot \frac{\partial \sigma}{\partial x'} \right\}_{x=x_i} dx',$$

or, putting

$$\frac{ds(x)}{dx} = \chi(x),$$

to

$$L_{i1} = \chi(x_i) \cdot \Delta x_i \cdot \{ \sigma(x_i, x'_1) - \sigma(x_i, x'_0) \} = \chi(x_i) \cdot \Delta x_i \cdot \sigma_{i1}.$$

Integrating along this same strip over x' from x'_0 to x'_n , i.e. from $-\infty$ to $+\infty$, we obtain

$$L_{in} = \chi(x_i) \cdot \Delta x_i \cdot 1.$$

So the quantity σ_{i1} , (i.e. the value of the function $\sigma(x, x')$ at $x = x_i$ and $x' = x'_1$) is found to be the quotient $\frac{L_{i1}}{L_{in}}$, whence

$$\sigma_{i1} = \frac{\Delta x_i \cdot \int_{-\infty}^{x'_1} D(x_i, x') dx'}{\Delta x_i \cdot \int_{-\infty}^{+\infty} D(x_i, x') dx'}.$$

We may find an approximate value of σ_{il} by taking a strip of finite breadth Δx_i round $x = x_i$.

As we only have at our disposal strips round $x = \xi_k = x_{k-1/2}$ having the breadth c , we are able to determine only approximately the value of $\sigma_{k-1/2, l} = \sigma(\xi_k, x'_l)$ by the formula

$$\sigma_{k-1/2, l} \approx \frac{c \cdot \int_{-\infty}^{x'_l} D(\xi_k, x') dx'}{c \cdot \int_{-\infty}^{+\infty} D(\xi_k, x') dx'} \quad \dots \quad (15)$$

where \approx signifies: approximately equal to.

The numerator of the fraction (15) equals the N^{th} part of the frequency content of the row of ξ_k from the lower limit up to the class-limit x'_l . The denominator is the N^{th} part of the frequency content of the whole row of ξ_k . Hence

$$\sigma_{k-1/2, l} \approx \frac{\sum_{j=1}^l Y_{kj}}{\sum_{j=1}^{n'} Y_{kj}} \quad \dots \quad (16)$$

In this way we find the value of the function $\sigma(x, x')$ belonging to each pair $(\xi_k, x'_l) = (x_{k-1/2}, x'_l)$.

By means of the relation $\sigma = \Theta(\zeta)$ we assign to each value of σ a value of ζ . Thus we obtain by each combination $(\xi_k, x'_l) = (x'_{k-1/2}, x'_l)$ a value $\zeta_{k-1/2, l}$.

If the correlation is really linear, ζ must appear to satisfy the relation

$$\begin{aligned} \zeta_{k-1/2, l} = t'_l - \gamma t_{k-1/2} &= h'u'_l - \gamma hu_{k-1/2} = h'(x'_l - \bar{\xi}') - \gamma h(\xi_k - \bar{\xi}) = \\ &= h'(x'_l - \bar{\xi}') - \gamma h(x_{k-1/2} - \bar{\xi}) = ax_{k-1/2} + \bar{a}x'_l + \beta. \end{aligned}$$

Inversely, if ζ satisfies the equation $\zeta = ax + \bar{a}x' + \beta$, it furnishes a second indication that the correlation is linear. From the 5 coefficients $a, b, \bar{a}, \bar{\beta}$ the 5 quantities $\bar{\xi}, \bar{\xi}', h, h', \gamma$ may be computed.

In the same manner we may analyze the frequency scheme by means of the quantities s' and σ' (resp. z' and ζ'). Then z' and $s' = \Theta(z')$ are functions only of x' ; ζ' and $\sigma' = \Theta(\zeta')$, on the contrary, are functions of both x and x' .

The value of $s'_l = s'(x'_l)$ is to be determined by

$$s_l = \frac{\sum_{i=1}^n \sum_{j=1}^l Y_{ij}}{N} \quad \dots \quad (14')$$

The numerator is the total frequency content of the columns belonging to $\xi'_1, \xi'_2, \dots, \xi'_l$. So we obtain $n' - 1$ pairs (x', s') ; hence by means of

$s' = \Theta(z')$ also $n'-1$ pairs (x', z') , $x'_0 (= -\infty)$ being conjugated to $s'_0 = 0$, $z'_0 = -\infty$ and $x'_{n'} (= +\infty)$ to $s'_{n'} = 1$, $z'_{n'} = +\infty$.

Besides we may compute the values

$$\sigma'(x_k, \xi'_l) = \sigma'(x_k, x'_{l-1/2}) = \sigma'_{k, l-1/2}$$

of the function $\sigma'(x, x')$ by the formula

$$\sigma'_{k, l-1/2} = \frac{\sum_{i=1}^k Y_{il}}{\sum_{i=1}^n Y_{il}} \quad \dots \quad (16')$$

From $\sigma' = \Theta(\xi')$ we find the values of the function ξ' at the points $x = x_k$, $x' = \xi'_l = x'_{l-1/2}$.

In applying this calculus in the case of linear correlation, z' must appear to satisfy a linear relation $z' = a'x' + b'$, ξ' satisfying a linear relation $\xi' = \bar{a}'x + a'x' + \beta'$.

Inversely, if it has been found, that the 4 linear relations $z = ax + b$, $z' = a'x' + b'$, $\xi = ax + a'x' + \beta$, $\xi' = \bar{a}'x + a'x' + \beta'$ are satisfied, we may conclude that the correlation between x and x' is linear indeed.

We have thus deduced a set of characteristics for linear correlation.

If we wish to know the value of the functions σ and ξ (resp. σ' and ξ') also in the pairs of class-limits $(x_k, x'_l) \rightarrow$ i.e. at the angular points of the frequency frame — we must determine the value of σ_{kl} (resp. ξ_{kl}) between $\sigma_{k-1/2, l}$ and $\sigma_{k+1/2, l}$ (resp. $\xi_{k-1/2, l}$ and $\xi_{k+1/2, l}$) by interpolation. Similarly we may compute the value of the functions σ' and ξ' at the points (x_k, x'_l) by interpolation. Then we know of both functions ξ and ξ' — be it approximately — the values in the network of the „points” (x_k, x'_l) .

The method by which the functions z, z', ξ and ξ' have been constructed in the above treatment of linear correlation, appears to be wholly analogous to the way in which the function z has been constructed in the case of non-normal frequency distribution of only one variable (see page 803 (7)). We shall therefore also follow this method in case the correlation is not linear. Then the only difference is that the functions $z(x)$, $\xi(x, x')$, $z'(x')$, $\xi'(x, x')$ are no more altogether linear.

In applying this method we follow the principle that — limiting ourselves to the correspondence between x and z — corresponding values of x and z are equally probable (cf. Skew Frequency Curves in Biology and Statistics, 2nd paper, p. 37).

Moreover we choose for the quantity z , normally distributed, a continuous, univalent, ever increasing function of x , so its lowest value the $(-\infty)$ corresponds to the lower limit of x , its highest value $(+\infty)$ to the upper limit of x (cf. S. F. C. i. B. a. S. 2nd paper, pp. 37, 38). These latter suppositions concerning continuity, univalency, and monotony are

only introduced for convenience' sake. They may be replaced by others according to circumstances (see: "Skew Frequency Curves", Proceed. of the Kon. Akad. v. Wet., Vol. XIX (1916) p. 670).

A similar reasoning may be held with regard to z' as a function of x' , to ζ so far as it is a function of x' , and to ζ' so far as it is a function of x .

Let us now summarize the course of the different operations:

Ia. Add the frequencies in each of the rows $\xi_1, \xi_2, \dots, \xi_k, \dots, \xi_n$:

$$R_k = \sum_{j=1}^{n'} Y_{kj} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

b. Add the frequencies in all the rows $\xi_1, \xi_2, \dots, \xi_k$ together:

$$S_k = R_1 + R_2 + \dots + R_k = \sum_{i=1}^k R_i = \sum_{i=1}^k \sum_{j=1}^{n'} Y_{ij}, \quad . \quad . \quad . \quad (18)$$

whence

$$N = S_n = \sum_{i=1}^n R_i;$$

c. then $s_k = s(x_k)$ results from

$$s_k = \frac{S_k}{N}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

d. and z_k from

$$s_k = \Theta(z_k) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{z_k} e^{-v^2} dv.$$

e. Add only the frequencies $Y_{k1}, Y_{k2}, \dots, Y_{kl}$ in the row ξ_k , which correspond to $\xi'_1, \xi'_2, \dots, \xi'_l$; hence

$$P_{k,l} = \sum_{j=1}^l Y_{kj}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

f. then $\sigma_{k-1/2,l} = \sigma(x_{k-1/2}, x'_l) = \sigma(\xi'_k, x'_l)$ is derived from

$$\sigma_{k-1/2,l} = \frac{P_{k,l}}{R_k}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

g. and $\zeta_{k-1/2,l}$ from

$$\sigma_{k-1/2,l} = \Theta(\zeta_{k-1/2,l}).$$

h. Finally ζ_{kl} is determined by interpolation.

II. Likewise with z' and ζ' :

a'. Add the frequencies in each of the columns $\xi'_1, \xi'_2, \dots, \xi'_l, \dots, \xi'_{n'}$:

$$R'_l = \sum_{i=1}^n Y_{il} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17')$$

b'. Add the frequencies in all the columns $\xi'_1, \xi'_2, \dots, \xi'_l$ together:

$$S'_l = R'_1 + R'_2 + \dots + R'_l = \sum_{j=1}^l R'_j = \sum_{i=1}^n \sum_{j=1}^l Y_{ij}, \quad . \quad . \quad (18')$$

We have thus got two sets of linearly correlated variables, viz. (z, ζ) and (z', ζ') , but the coefficient of correlation is zero; properly speaking there is no correlation at all between them. We have solved the problem, it is true, but this solution is by no means satisfactory. In the above discussion about the requirements of a useful solution, we have already explained why we do not remain by the two sets of variables in question.

In future we shall introduce z and z' — in the place of x and x' — as fundamental variables. Then ζ and ζ' become, in general, functions both of z and z' . Thus the two-dimensional differentials $dz \cdot d\zeta$ and $dz' \cdot d\zeta'$ will be transformed in this way:

$$dz \cdot d\zeta \rightleftharpoons dz \cdot \frac{\partial \zeta}{\partial z'} \cdot dz', \quad dz' \cdot d\zeta' \rightleftharpoons dz' \cdot \frac{\partial \zeta'}{\partial z} \cdot dz.$$

So we find for dW :

$$dW = \frac{1}{\pi} e^{-(z^2 + \zeta^2)} \frac{\partial \zeta}{\partial z'} \cdot dz \cdot dz' = \frac{1}{\pi} e^{-(z'^2 + \zeta'^2)} \frac{\partial \zeta'}{\partial z} \cdot dz \cdot dz', \quad (23)$$

whence

$$e^{-(z^2 + \zeta^2)} \frac{\partial \zeta}{\partial z'} = e^{-(z'^2 + \zeta'^2)} \frac{\partial \zeta'}{\partial z} \quad (24)$$

We next put

$$z^2 + \zeta^2 = r^2, \quad z'^2 + \zeta'^2 = r'^2 \quad (25)$$

In order to orient ourselves in the treatment, we shall previously make a simplifying supposition, viz. that at each point (x, x') or (z, z') holds

$$r' = r, \quad (26)$$

whence, on account of (24), we have also at each point

$$\frac{\partial \zeta}{\partial z'} = \frac{\partial \zeta'}{\partial z} \quad (27)$$

The result of computing the values of the functions z, ζ, z', ζ' must make out whether the expressions $r^2 = z^2 + \zeta^2$ and $r'^2 = z'^2 + \zeta'^2$ are equal at each point, or at least nearly equal. We may not expect a complete empirical concordance between r^2 and r'^2 , if even it did exist theoretically, since the computed values of ζ and ζ' are exposed to inevitable inaccuracies. Especially on the borders of the frequency domain we must be prepared for a bad concordance between the theoretical and the empirical values.

We shall next make use of two rectangular systems of coordinates: (z, ζ) and (z', ζ') , having the same origin.

The way in which the functions z, z', ζ, ζ' are introduced, viz. as argument of the Θ -function, points out, that they all are permanently increasing functions, resp. of s, s', σ, σ' . Moreover, by ranging the values of x to ascending magnitude and by adding the successive frequencies,

The quantities A, B are, as functions of z, z', ξ, ξ' , in the last instance, functions of x and x' . So each pair (A, B) corresponds to a pair (x, x') . Now the pairs (x, x') are not equally probable. These different probabilities of (x, x') will come out at different weights of the pairs (A, B) . The pairs (x, x') , for which the values of A and B are known, refer to the class-limits, to the angular points of the frequency frame. But the chances, or the frequencies Y , proportional to them, are given empirically in the combinations $(\xi_k, \xi'_l) = (x_{k-1/2}, x'_{l-1/2})$. We may now take as the frequency of the pair (x_k, x'_l) the average of the frequencies belonging to the four surrounding class-centres:

$$(\xi_k, \xi'_l) = (x_{k-1/2}, x'_{l-1/2}), (\xi_k, \xi'_{l+1}) = (x_{k-1/2}, x'_{l+1/2}), (\xi_{k+1}, \xi'_l) = (x_{k+1/2}, x'_{l-1/2}), (\xi_{k+1}, \xi'_{l+1}) = (x_{k+1/2}, x'_{l+1/2}).$$

So we take as the frequency of (x_k, x'_l) :

$$\frac{1}{4} (Y_{kl} + Y_{k,l+1} + Y_{k+1,l} + Y_{k+1,l+1}).$$

Hence the weight of the combination (k, l) , which is proportional to the frequency of (x_k, x'_l) may be equalled to

$$G_{kl} = Y_{kl} + Y_{k,l+1} + Y_{k+1,l} + Y_{k+1,l+1} \quad , \quad \sum \sum G_{kl} = 4N.$$

So we have several points (A_k, B_l) with weights G_{kl} and we want to fit to them the line $B = A.M$ passing through the origin.

This problem does not properly differ essentially from that of linear correlation. Indeed, given N combinations (ξ_k, ξ'_l) with frequencies Y_{kl} , according to the theory of linear correlation, the combination of the arithmetic means $(\bar{\xi}, \bar{\xi}')$ is the most probable one.

$u_k = \xi_k - \bar{\xi}, u'_l = \xi'_l - \bar{\xi}'$ being the deviations from the resp. means, the combination (u_k, u'_l) may be represented by a point with rectangular (rectilinear) coordinates u_k, u'_l ; then the points (u_k, u'_l) , which have equal probability, lie on an ellipse

$$h^2 u^2 - 2\gamma h h' u u' + h'^2 u'^2 = \text{constant},$$

where h, h' and γ are constants.

The straight line, best adapted to all the points, appears to be the common major axis of all these ellipses. The angle φ , which the major axis makes with the axis of u , is determined by

$$\text{tg } 2\varphi = \frac{-2\gamma h h'}{h^2 - h'^2}.$$

We have now:

$$\mu = \overline{u^2} = \frac{1}{2h^2(1-\gamma^2)} \quad , \quad \mu' = \overline{u'^2} = \frac{1}{2h'^2(1-\gamma^2)} \quad , \quad \lambda = \overline{u u'} = \frac{\gamma}{2hh'(1-\gamma^2)},$$

whence

$$\mu : \mu' : \lambda = h'^2 : h^2 : \gamma h h'.$$

So we obtain

$$\operatorname{tg} 2\varphi = \frac{2\lambda}{\mu - \mu'}.$$

Applying this to the present case, we have

$$\mu = \overline{A^2} = \frac{\sum G_{kl} A_{kl}^2}{4N}, \quad \mu' = \overline{B^2} = \frac{\sum G_{kl} B_{kl}^2}{4N}, \quad \lambda = \overline{AB} = \frac{\sum G_{kl} A_{kl} B_{kl}}{4N},$$

whence

$$\operatorname{tg} 2\varphi = \frac{2AB}{A^2 - B^2}.$$

Incidentally we shall also compute the mean error of the value of φ , obtained by the above adjustment.

From $\operatorname{tg} 2\varphi = \frac{2\lambda}{\mu - \mu'}$, results

$$\sec^2 2\varphi \cdot \Delta \varphi = \frac{(\mu - \mu') \Delta \lambda - \lambda \Delta \mu + \lambda \Delta \mu'}{(\mu - \mu')^2},$$

so that

$$\sec^4 2\varphi \cdot \overline{\Delta \varphi^2} =$$

$$\frac{(\mu - \mu')^2 \Delta \lambda^2 - 2\lambda(\mu - \mu') \Delta \lambda \Delta \mu + 2\lambda(\mu - \mu') \Delta \lambda \Delta \mu' + \lambda^2 \Delta \mu^2 - 2\lambda^2 \Delta \mu \Delta \mu' + \lambda^2 \Delta \mu'^2}{(\mu - \mu')^4}.$$

We now have

$$\Delta \lambda^2 = \frac{\lambda^2 + \mu\mu'}{N}, \quad \Delta \lambda \cdot \Delta \mu = \frac{2\lambda\mu}{N}, \quad \Delta \lambda \cdot \Delta \mu' = \frac{2\lambda\mu'}{N},$$

$$\Delta \mu^2 = \frac{2\mu^2}{N}, \quad \Delta \mu \cdot \Delta \mu' = \frac{2\lambda^2}{N}, \quad \Delta \mu'^2 = \frac{2\mu'^2}{N},$$

therefore

$$\sec^4 2\varphi \cdot \Delta \varphi^2 = \frac{(\mu\mu' - \lambda^2) \{(\mu - \mu')^2 + 4\lambda^2\}}{N(\mu - \mu')^4}$$

or, since

$$\sec^2 2\varphi = 1 + \operatorname{tg}^2 2\varphi = \frac{(\mu - \mu')^2 + 4\lambda^2}{(\mu - \mu')^2},$$

$$\varepsilon_{\varphi}^2 = \Delta \varphi^2 = \frac{1}{N} \frac{\mu\mu' - \lambda^2}{(\mu - \mu')^2 + 4\lambda^2},$$

so, in the present case:

$$\varepsilon_{\varphi}^2 = \frac{1}{N} \frac{A^2 \cdot B^2 - AB^2}{(A^2 - B^2)^2 + 4AB^2}.$$

The tangent M of the angle φ , included between the major axis and the A -axis, is the most probable value of the fraction $\frac{B}{A}$. So M too is the most probable value of $\operatorname{tg} \omega$, whence φ is the most probable value $\bar{\omega}$ of ω .

Thus we must compute this most probable (mean) value $\bar{\omega}$ of ω from

$$\operatorname{tg} 2 \bar{\omega} = \frac{2 \overline{AB}}{\overline{A^2 - B^2}} \quad \text{with} \quad \varepsilon_{\bar{\omega}}^2 = \frac{1}{N} \frac{\overline{A^2} \cdot \overline{B^2} - \overline{AB}^2}{(\overline{A^2 - B^2})^2 + 4 \overline{AB}^2} \quad (35)$$

where

$$A = z z' - \zeta \zeta' \quad , \quad B = z \zeta' + z' \zeta.$$

It is now our object to find the mean (i.e. the most probable) value of the coefficient of correlation $\gamma = \cos \omega$. We choose for this

$$\bar{\gamma} = \cos \bar{\omega}, \quad (36)$$

its mean error being

$$\varepsilon_{\bar{\gamma}} = \sin \bar{\omega} \cdot \varepsilon_{\bar{\omega}} \quad (37)$$

So, our first result, based upon the simplest suppositions, runs:

If, for each combination (z_k, z'_l) , the expression $r^2 = z^2 + \zeta^2$ is found to be practically equal to $r'^2 = z'^2 + \zeta'^2$ and if ω is found to be tolerable constant, then, in solving the problem, we may choose for t and t' :

$$t = \frac{z}{\sin \omega} \quad , \quad t' = \frac{z'}{\sin \omega} \quad , \quad \gamma = \cos \omega$$

where each of the final variables appears to be a function of only one of the primary variables, e.g. t of x , and t' of x' .

If, the relation $r^2 = r'^2$ being satisfied, ω is not a constant, and so depends on $r (=r')$, then in

$$z^2 - 2 \cos \omega \cdot z z' + z'^2 = r^2 \sin^2 \omega$$

ω is variable together with r .

Then the ellipses $r = \text{const.}$ ($\omega = \text{const.}$) are no longer homothetic. We, however, may transform them into a set of homothetic ellipses, in the following way:

We turn the coordinate system (z, z') 45° , so that the axes of the new coordinates u and v coincide with the axes of symmetry. Accordingly putting

$$z = \frac{u-v}{\sqrt{2}} \quad , \quad z' = \frac{u+v}{\sqrt{2}},$$

we obtain

$$\begin{aligned} z^2 - 2 \cos \omega \cdot z z' + z'^2 &= \frac{(u-v)^2 - 2 \cos \omega \cdot (u^2 - v^2) + (u+v)^2}{2} = \\ &= (1 - \cos \omega) u^2 + (1 + \cos \omega) v^2 = r^2 \sin^2 \omega \end{aligned}$$

or

$$\frac{u^2}{1 + \cos \omega} + \frac{v^2}{1 - \cos \omega} = r^2.$$

Now, introducing a constant angle Ω , we put

$$u = U \sqrt{\frac{1 + \cos \omega}{1 + \cos \Omega}}, \quad v = V \sqrt{\frac{1 - \cos \omega}{1 - \cos \Omega}};$$

and so find

$$\frac{U^2}{1 + \cos \Omega} + \frac{V^2}{1 - \cos \Omega} = r^2.$$

By turning back the coordinate axes into their former position, by means of the transformation

$$U = \frac{Z + Z'}{\sqrt{2}}, \quad V = \frac{-Z + Z'}{\sqrt{2}},$$

we obtain the equation

$$Z^2 - 2 \cos \Omega \cdot Z Z' + Z'^2 = r^2 \sin^2 \Omega,$$

after which the substitution

$$Z = T \sin \Omega, \quad Z' = T' \sin \Omega$$

leads to

$$T^2 - 2 \cos \Omega \cdot T T' + T'^2 = r^2.$$

Finally, the variables T and T' depend on z, z' (resp. t, t') according to the following formula:

$$\begin{aligned} T = \frac{Z}{\sin \Omega} = \frac{U - V}{\sin \Omega \cdot \sqrt{2}} &= \frac{\sqrt{\frac{1 + \cos \Omega}{1 + \cos \omega}} \cdot u - \sqrt{\frac{1 - \cos \Omega}{1 - \cos \omega}} \cdot v}{\sin \Omega \cdot \sqrt{2}} = \\ &= \frac{\cos \frac{\Omega}{2}}{\cos \frac{\omega}{2}} (z + z') - \frac{\sin \frac{\Omega}{2}}{\sin \frac{\omega}{2}} (-z + z') \\ &= \frac{2 \sin \Omega}{2 \sin \Omega} \end{aligned}$$

$$\begin{aligned} T' = \frac{Z'}{\sin \Omega} = \frac{U + V}{\sin \Omega \cdot \sqrt{2}} &= \frac{\sqrt{\frac{1 + \cos \Omega}{1 + \cos \omega}} \cdot u + \sqrt{\frac{1 - \cos \Omega}{1 - \cos \omega}} \cdot v}{\sin \Omega \cdot \sqrt{2}} = \\ &= \frac{\cos \frac{\Omega}{2}}{\cos \frac{\omega}{2}} (z + z') + \frac{\sin \frac{\Omega}{2}}{\sin \frac{\omega}{2}} (-z + z') \\ &= \frac{2 \sin \Omega}{2 \sin \Omega} \end{aligned}$$

or

$$\left. \begin{aligned} T &= \frac{\sin \frac{\omega + \Omega}{2} \cdot z + \sin \frac{\omega - \Omega}{2} \cdot z'}{\sin \omega \cdot \sin \Omega} = \frac{\sin \frac{\omega + \Omega}{2} \cdot t + \sin \frac{\omega - \Omega}{2} \cdot t'}{\sin \Omega} \\ T' &= \frac{\sin \frac{\omega - \Omega}{2} \cdot z + \sin \frac{\omega + \Omega}{2} \cdot z'}{\sin \omega \cdot \sin \Omega} = \frac{\sin \frac{\omega - \Omega}{2} \cdot t + \sin \frac{\omega + \Omega}{2} \cdot t'}{\sin \Omega} \end{aligned} \right\} \quad (38)$$

Theoretically we are wholly free in choosing Ω . If, however, we try to relate T as closely as possible to t (and z), and T' to t' (and z'), then we shall dispose of Ω in such a way that the average value of $\omega - \Omega$ is as small as possible. This may be attained by taking for Ω the mean value $\bar{\omega}$ of ω . This mean may be computed:

either empirically in the same way as if ω is practically constant (see (35))

or theoretically, if we have a clear idea about the functional relation existing between ω and r . When, in this case, $\omega = f(r)$, then

$$\bar{\omega} = \int_0^{\infty} f(r) dW = \int_0^{\infty} f(r) r e^{-r^2} dr.$$

Thus we go on with

$$\left. \begin{aligned} T &= \frac{\sin \frac{\omega + \bar{\omega}}{2} \cdot z + \sin \frac{\omega - \bar{\omega}}{2} \cdot z'}{\sin \omega \cdot \sin \bar{\omega}} = \frac{\sin \frac{\omega + \bar{\omega}}{2} \cdot t + \sin \frac{\omega - \bar{\omega}}{2} \cdot t'}{\sin \bar{\omega}} \\ T' &= \frac{\sin \frac{\omega - \bar{\omega}}{2} \cdot z + \sin \frac{\omega + \bar{\omega}}{2} \cdot z'}{\sin \omega \cdot \sin \bar{\omega}} = \frac{\sin \frac{\omega - \bar{\omega}}{2} \cdot t + \sin \frac{\omega + \bar{\omega}}{2} \cdot t'}{\sin \bar{\omega}} \end{aligned} \right\} \quad (39)$$

together with

$$T^2 - 2\gamma T T' + T'^2 = r^2, \quad \gamma = \cos \bar{\omega}.$$

The variables T and T' may now be submitted to the transformation (9):

$$\begin{aligned} t &= \frac{\sin(\tau + \bar{\omega})}{\sin \bar{\omega}} T - \frac{\sin \tau}{\sin \bar{\omega}} T' \\ t' &= \frac{\sin \tau}{\sin \bar{\omega}} T - \frac{\sin(\tau - \bar{\omega})}{\sin \bar{\omega}} T'. \end{aligned}$$

After reduction we obtain

$$\left. \begin{aligned} t &= \frac{\sin\left(\frac{\omega + \bar{\omega}}{2} + \tau\right)}{\sin \bar{\omega}} t + \frac{\sin\left(\frac{\omega - \bar{\omega}}{2} - \tau\right)}{\sin \bar{\omega}} t', \\ t' &= \frac{\sin\left(\frac{\omega - \bar{\omega}}{2} + \tau\right)}{\sin \bar{\omega}} t + \frac{\sin\left(\frac{\omega + \bar{\omega}}{2} - \tau\right)}{\sin \bar{\omega}} t' \end{aligned} \right\} \quad (40)$$

or

$$\begin{aligned} t &= \frac{\sin\left(\frac{\omega + \bar{\omega}}{2} + \tau\right)}{\sin \omega \cdot \sin \bar{\omega}} z + \frac{\sin\left(\frac{\omega - \bar{\omega}}{2} - \tau\right)}{\sin \omega \cdot \sin \bar{\omega}} z', \\ t' &= \frac{\sin\left(\frac{\omega - \bar{\omega}}{2} + \tau\right)}{\sin \omega \cdot \sin \bar{\omega}} z + \frac{\sin\left(\frac{\omega + \bar{\omega}}{2} - \tau\right)}{\sin \omega \cdot \sin \bar{\omega}} z' \end{aligned} \quad (41)$$

We may still freely dispose of τ . It is not necessary that τ should be a constant; it may be a function of z and z' or only of r .

We shall choose τ in such a way that the expression (41) for t gets rid of the term with z' . For this we evidently must put

$$\tau = \frac{\omega - \bar{\omega}}{2};$$

then we obtain

$$\begin{aligned} t &= \frac{z}{\sin \omega} \\ t' &= \frac{\sin(\omega - \bar{\omega})}{\sin \omega \cdot \sin \bar{\omega}} \cdot z + \frac{1}{\sin \bar{\omega}} z' \end{aligned} \quad (42)$$

So doing we have succeeded in making one of the variables (t) into a pure function of z , i. e. of x . But then the other variable is a function both of z and z' (thus also both of x and x').

Similarly it would have been possible, by a suitable choice of t , to make the variable t into a function only of z' (i. e. only of x'), or the variable t' into a function only of z (i. e. only of x), or into a function only of z' (i. e. only of x'). So our second result runs:

If, for each combination (z_k, z'_l) , the expression $r^2 = z^2 + \bar{z}^2$ is found almost equal to $r'^2 = z'^2 + \bar{z}'^2$, but if the condition that in each combination (z_k, z'_l) ω has the same value, is not fulfilled, then it is possible to take for one of the final variables (for instance t) a function of only one of the primary variables (for instance x), the other final variable (t') remaining a mixed function of x and x' (see (42)). In this case there exists a functional relation between ω and r .

Further on we shall give some hints for finding out the function $\omega = f(r)$.

(To be continued).

Chemistry. — ERNST COHEN and SABURO MIYAKE: "*The influence of minute traces of Water on Solution Equilibria.*" II

(Communicated at the meeting of October 31, 1925).

Introductory.

In the first paper under the above title ¹⁾ it was shown that the solution equilibrium between salicylic acid and benzene is shifted by the presence of minute traces of water, the solubility of salicylic acid markedly increasing by the addition of water. We also stated that the phenomenon in question can manifest itself in consequence of the presence of occluded water, so that, if the solubility of substances must be determined with accuracy, great care should be taken to work with pure (dried) substances. At the time we pointed out that the minute quantity of occluded water can be accurately determined by solubility determinations. The experiments which are described in this paper, have been made in order to become acquainted with the influence of minute traces of water in other solution equilibria, and moreover — the measurements which refer to this part of the problem will be described first — to prove, more directly than was done in the first paper, that the phenomena which are investigated here, are indeed (true) equilibria.

As the method of procedure was fully explained in the first communication we need not go into details here.

All the solubility determinations have been carried out at 30°.50 C.

1. *The Equilibrium in the System Salicylic acid — Benzene.*

1. We used the salicylic acid (preparation KAHLBAUM "für kalorimetrische Bestimmungen", vide our first communication under A. § 1a), which, according to the method described there (B. § 2. 2) was recrystallized from dry ether (vide first communication under A. § 1. c). The acid had been kept for a long time in vacuo over P_2O_5 , and had been powdered every day until the smell of ether had totally disappeared.

2. The benzene (free from thiophene) had been treated in the way previously described (vide first communication under A. § 1. b.). Whereas in our first investigation we simply poured the liquid out of the flask into the shaking bottles, so that it came in contact with the moist air in the laboratory, we now pressed it from the flask by means of air which had been carefully dried over sulphuric acid and then over P_2O_5 . After having used the apparatus for a month we ascertained that practically the benzene

¹⁾ These Proceedings 28, 702 (1925); Zeitschr. f. physik. Chemie 118, 37, (1925).

had remained unchanged: the solubility of the salicylic acid which had been determined at 1.00; 1.02; 1.01; 1.02 *), was found to be 1.02 at the end of that period (vide first communication § 13).

3. In order to prove that the phenomena, described in the first communication, are indeed equilibria, we set to work as follows: In each of two shaking bottles about one gramme of dry acid and about 40 gms. of dry benzene is weighed; then these bottles are sealed. One of the bottles is immediately put into the thermostat, the temperature of which has been set to the temperature of the experiment ($30^{\circ}.50$ C.). The second bottle, with its contents, is heated in a water bath to such a temperature that the solute has been almost, but not wholly, dissolved. Then this bottle is also placed in the thermostat, and both bottles are shaken for the same length of time, at the temperature of the experiment.

The same manipulations are carried out with two bottles, the contents of which differ from those mentioned above in that they contain moist benzene (of an accurately determined water percentage) instead of dry benzene.

In this way the final condition is reached in both cases (dry and moist benzene), as well from the side of supersaturation as from that of undersaturation.

TABLE 1.
Solubility of Salicylic acid in Benzene.
Temperature $30^{\circ}.50$ C.

Number of the experiment	Period of shaking in hours	Thousandths of weight $\frac{0}{0}$ of water	Solubility in weight $\frac{0}{0}$
1	4	0	1.00 1.02*
2	3	0	1.01 1.02*
3	3	66	1.16 1.17*
4	4	71	1.19 1.18*

As we see from Table 1 we find identical values for the solubilities, within the errors of the experiment, in other words: in the dry system, as well as in the wet, we have indeed true equilibria.

Weight % means gms. of solute in 100 gms. of solution.

2. *The Equilibrium in the System Salicylic acid — Chloroform.*

4. When the chloroform had been for a long time on P_2O_5 , it was distilled from fresh P_2O_5 , in the way previously described (vide first

*) Means that equilibrium is reached from the side of supersaturation.

communication A. § 1b.). The tapping from the flask was done as described in § 2.

When tapping the saturated solutions of salicylic acid we placed the bulb E (Fig. 1 first communication) always in ice water, so as to keep the tension of the vapour over the solution as low as possible.

The following figures prove that the analysis by evaporation of the solvent (vide first communication § 5) gives good results, whether dry chloroform is taken or moist: 0.2631 gms. of dry salicylic acid dissolved in about 17 gms. of dry chloroform gave 0.2629 gms. after the solvent was evaporated. When the residue had been in vacuo over sulphuric acid and salicylic acid for the night, the weight proved to be 0.2629 gms. Further 0.3681 gms. of the acid, which had been dissolved in about 20 gms. of moist chloroform (water-content of the chloroform 0.0354 gms. of water in 100 gms. of chloroform) gave 0.3676 gms. after being treated in the same way.

Table 2 contains the results of the experiments with chloroform.

TABLE 2.
Solubility of salicylic acid in Chloroform.
Temperature 30°.50 C.

Number of the experiment	Period of shaking in hours	Thousandths of weight % of water	Solubility in weight %
5	3	0	1.55 1.56*
6	5	0	1.56 1.55*
7	3	35.4	1.64 1.63*
8	3	60.8	1.69 1.68*
9	3	108.1	1.72 1.71*
10	3	saturated	1.73 1.72*

For further elucidation of the term "saturated" in this table and the following ones, we wish to observe that the solvent had previously been shaken with excess of water at 30°.50 C. and after separating it from the excess of water was used for the determination of solubility.

3. *The Equilibrium in the System Salicylic acid — Carbon Tetrachloride.*

5. The carbon tetrachloride we used was free from sulphur; when it had been on P_2O_5 for a week, it was distilled from a fresh quantity of P_2O_5 . The method of keeping it and of transferring it to the shaking bottles is described in § 2.

In this case too the analysis of the saturated solutions can be made by evaporation of the solvent, as may be seen from the following figures:

0.0432 gms. of salicylic acid, dissolved in about 15 gms. of CCl_4 , gave, after evaporation of the solvent, 0.0426 gms. of acid. In the second experiment we weighed 0.0321 gms. into the bottle and found back 0.0321 gms.

Table 3 gives a summary of the results.

TABLE 3.
Solubility of Salicylic acid in Carbon tetrachloride.
Temperature $30^\circ.50$ C.

Number of the experiment	Period of shaking in hours	Thousandths of weight % of water	Solubility in weight %	
11	3	0	0.36	0.35*
12	5	0	0.35	0.35*
13	3	8.7	0.35	0.36*
14	3	22.7	0.36	0.36*
15	3	65.9	0.36	0.37*
16	3	saturated	0.36	0.36*

4. *The Equilibrium in the System o-Nitrobenzoic acid — Benzene.*

6. With a view to some experiments on the metastability of this acid as a consequence of enantiotropy or monotropy, we had acquired a rather large quantity. We used preparations supplied by different factories. After having been recrystallized from water they gave identical results. This was seen f.i. when determining their solubility in water.

TABLE 4.
Solubility of o-nitrobenzoic acid in Benzene.
Temperature $30^\circ.50$ C.

Number of the experiment	Period of shaking in hours	Thousandths of weight % of water	Solubility in weight %	
17	3	0	0.35	0.35*
18	3	37.7	0.43	0.43*
19	3	66.3	0.49	0.49*
20	3	89.1	0.49	0.50*
21	3	saturated	0.50	0.50*

Also in this case we found the evaporation method, when determining the solubility, very useful. Thus 0.0327 gms. of acid, dissolved in 15 gms.

of moist benzene (water content 0.2226 gms. of water in 100 gms. of benzene) gave 0.0327 gms. of acid after evaporation of the solvent. When the residue had been for 24 hours in vacuo over P_2O_5 and *o*-nitrobenzoic acid, we found 0.0327 gms. In a second experiment in which 0.0558 gms. had been weighed into the bottle, we found in the same way 0.0559 gms.

Table 4 contains the results of these measurements.

5. *The Equilibrium in the System o-Nitrobenzoic acid — Chloroform.*

7. As the experiments in this case were carried out in exactly the same way as in the system salicylic acid—chloroform, it will suffice to point out that the evaporation process yields accurate results. Thus, after weighing 0.0906 gms. of acid into the bottle, we found 0.0907 gms. after evaporation. In another experiment 0.0414 gms. instead of 0.0418 gms.

TABLE 5.
Solubility of *o*-nitrobenzoic acid in Chloroform.
Temperature 30°.50 C.

Number of the experiment	Period of shaking in hours	Thousandths of weight % of water	Solubility in weight %	
22	3	0	0.45	0.45*
23	5	0	0.45	0.45*
24	3 $\frac{1}{4}$	25.6	0.51	0.52*
25	3	31.2	0.56	0.56*
26	3	saturated	0.55	0.56*
27	3	"	0.56	—

6. *The Equilibrium in the System Anthracene — Benzene.*

8. We have also investigated the influence of minute traces of water on the equilibrium between a non-electrolyte and benzene. We chose anthracene as a non-electrolyte.

A so-called pure preparation (resublimated) gave in the solubility determination in dry benzene (period of shaking 3 hours) the values of 2.05; 2.09. When the period of shaking was five hours we found 2.08; 2.09. When this preparation had been recrystallized from dry benzene we found 2.00; 2.00. To check the result we recrystallized this preparation from dry ether, left the mass thus obtained for some days in vacuo over P_2O_5 , and then determined the solubility again in dry benzene. Now we found 2.00; 2.00. We used this preparation for our final solubility determinations in benzene.

In this investigation we also made use of the evaporation method in

order to analyze the saturated solutions, when the following experiments had proved that, in this case too, accurate results were obtained; 0.1646 gms. of anthracene gave 0.1648 gms. after evaporation of the dry benzene in which they had been dissolved. In a second experiment we dissolved 0.1603 gms. of anthracene in moist benzene; we found 0.1605 gms.

TABLE 6.
Solubility of Anthracene in Benzene.
Temperature 30°.50 C.

Number of the experiment	Period of shaking in hours	Thousandths of weight % of water	Solubility in weight %	
28	3	0	2.00	2.00*
29	3	0	2.00	2.00*
30	3	38.9	2.00	2.00*
31	3	88.8	1.98	1.99*
32	3	saturated	1.98	1.98*

9. We shall not dwell upon the older literature about the solubility of the substances we investigated, as it refers to preparations, the purity (dryness) of which had not been closely examined. Therefore it is of no importance for the problem which occupies us here.

Summary.

Minute traces of water have great influence on solution equilibria in the systems salicylic acid — benzene, salicylic acid — chloroform, o-nitrobenzoic acid — benzene, o-nitrobenzoic acid — chloroform. No influence could be ascertained in the systems salicylic acid — carbon tetrachloride, and anthracene — benzene.

We intend to treat further conclusions in a subsequent communication.

VAN 'T HOFF-Laboratory.

Utrecht, September 1925.

Physiology. — G. GRIJNS : “Diet and reproduction.”

(Communicated at the meeting of October 31, 1925).

It is since long custom among breeders to feed animals, which are raised for reproductive purposes, along an other scheme as those, that will be used for working or butchering. Especially to much fattening is avoided in the first class. On a direct relation of nutrition and fertility textbooks give little or nothing.

It is only the last few years, since everywhere investigations on vitamins and on the adequacy of different albumens are going on, that this relation has been attended to and publications on this topic are growing more numerous. However they often differ very much, even they are sometimes contradictory.

HART, STEENBOCK and HUMPREY stated ¹⁾ that cows on a diet exclusively derived from the oat plant only bore dead young and such as died soon after birth. Addition of casein or A-vitamin did not ameliorate the condition, but calcium did. As oat is poor in calcium compounds, the cause of failure in reproduction was calcium starvation.

MCCOLLUM, in collaboration with SIMMONDS and PARSONS ²⁾ found, that rats on rations poor in albumen or containing an inadequate albumen reproduce badly and that starvation of A-vitamine leads to infertility.

The same thing recently was demonstrated by SIMMONNET ³⁾ who examined in extenso the influence on females and males separately.

Lack of watersoluble B also stops reproduction.

The failure of reproduction in these cases may be attributed to the fact, that, when food-substances that are necessary for building up the young bodies are absent, the young can not thrive and die from starvation. This may occur in the womb already, when either abortion or resorption of the foetus will take place ; or during lactation.

MATTILL experimented with CONKLIN ⁴⁾ and afterwards with STONE ⁵⁾ on the relation between nutrition and reproduction. They stated that rats fed on milk only grew well till about 90 days, there after they grew more slowly than rats on normal rations. Addition of yeast extract promotes growth, but in both cases they were infertile. Starch did not improve

¹⁾ Research Bullet. Agric. Exp. Station Madison 49, 1920.

²⁾ MC. COLLUM, N. SIMMONDS & H. T. PARSONS. Protein values of food. Journ. of Biol. Chem. XLVII, p. 111, & 139, 1921.

³⁾ H. SIMONNET. Influence de la carence en facteur lipo-soluble sur les fonctions de reproduction. Annales de Physiol. et de physicochimie biol. T. I. N^o. 3 p. 332, 1925.

⁴⁾ H. A. MATTILL & R. E. CONKLIN. The nutritive properties of milk. I. Journ. of Biol. Chem. XLIV p. 137. 1920.

⁵⁾ H. A. MATTILL & N. C. STONE. The nutritive properties of milk. II *ibid.* LV. p. 443. 1922.

reproduction. Out of 8 male rats, mated with females on normal diet 5 proved sterile. Supplementing milkpowder with brewersyeast gave some litters, but all died within two days.

Several investigators remark that a too high content of albumen damages the rearing of the young. ¹⁾

To quite different results ANDEREGG ²⁾ arrives. He fed rats on milkpowder supplemented with dextrin, agar 4 % (for bulk) and ferric citrate 0.2 %. On 70 % milkpowder and 26 % dextrin 23 out of 70 young were reared; on 60 % milkpowder with 36 % dextrin and a small amount of saltmixture growth was better than normal and reproduction normal.

8 females on a diet of milkpowder 60, casein 6, citrate of iron 0.2, agar 4 and dextrin 29.8 bore 129 young of which 75 or 59 % were reared. 9 females in whose ration 2.4 dextrin had been substituted by saltmixture bore 235 young of which 202 or 86 % were weaned, what equals normal nutrition.

When skimmed milkpowder was used instead of wholemilkpowder most females were infertile, the others having small litters.

A. also investigated the influence of wheatembryo on reproduction. He fed a mixture of casein 18, butterfat 5, saltmixture 3.7, wheatembryo from 1.5 to 9 and dextrin to make 100. All rats that got less than 6 % wheat embryo were sterile.

16 females on 7 % wheatembryo bore 79 young from which 32 or 29 % were reared. For 6 females on 8 % these figures were 65 and 59 or 91 % and 3 rats on 9 % wheatembryo raised all their 17 young.

ANDEREGG leaves undecided whether supplementing of lacking vitamin B, or of some other factor plays a part here. In my opinion A-vitamin might as well be a limiting factor here.

BARNETT SURE ³⁾ found rats on a diet chiefly composed of milkpowder, to which dextrin, saltmixture and water- and fat-soluble vitamins had been added, mostly sterile. Addition of different aminoacids did not improve this, so that the failure of reproduction can not be attributed to inadequate albumen. Fresh lettuce and velvet beans ⁴⁾ were able to conserve reproduction. Also when the milkpowder was fed with rice, with corn or with oat the animals bore young.

In a later experience ⁵⁾ he supplemented a ration of artificial nutriments,

¹⁾ SMITH. Journ. of Biol. Chem. XXVIII p. 215. 1917. HART & STEENBOCK. J. o. Biol. Chem. XXXIII p. 313. 1918. GLADYS HARTWELL. Bioch. Journ. XV p. 564 and ibid. XVI p. 78. 1922.

²⁾ L. F. ANDEREGG. Diet in relation to reproduction and rearing of the young. J. o. Biol. Chem. LIX p. 587. 1924.

³⁾ BARNETT SURE. Dietary requirements for reproduction I. J. o. Biol. Chem. LVIII p. 681. 1924.

Id. id. II J. o. Biol. Chem. LVIII p. 693.

⁴⁾ *Stizolobium deeringianum*. From Georgia and Southern States.

⁵⁾ BARNETT SURE. Dietary requirements for reproduction. III. J. o. Biol. Chem. LXII p. 371. 1925.

that contained sufficient A-, B-, and C-vitamin with several extracts and oils, withdrawing so much starch, that the caloric value remained constant. He then found, that cotton seed oil (5 %) or olive oil was apt to support fertility; the same did ethereal extracts from wheatembryo, yellow corn and hempseed. Coconut oil, linseed oil and sesam oil proved ineffective. He therefrom concludes to the existence of a reproduction controlling vitamin.

To the same conclusion come EVANS and BURR¹⁾. On a ration of albumen, fat, starch, saltmixture, yeast extract and cod liver oil their rats become infertile in the first or the second generation. In males this leads in a great deal of cases to testicular atrophy. Then it is irreparable. In females ovulation persists, but when fertilised by normal males, between the 12th and 20th day the foetus die and resorption occurs.

From wheatembryo they could extract with ether a substance, that was not destroyed by saponifying with 20 % alcoholic kalihydroxide. After a series of treatments with methylic alcohol, pentane, benzene etc. they arrive at a concentrated liquor, one drop of which a day is able to conserve fertility in rats on the above mentioned diet. They propose to name the active substance Vitamin E. This substance is present in small quantities in the intern organs, a little more in bodyfat and muscles. Animals on E-free diet are exempt of it. It is abundant in wheat especially in the embryo, also in green parts of plants and in a number of seeds and resistant to heating.

Against the experiments of ANDEREGG, who saw good reproduction on wholemilkpowder, they state, that by adding more fat infertility appears, what could be explained for by assuming that at a higher fat intake more vitamin E is wanted.

SURE²⁾ demonstrated the solubility of this fertilising factor in benzene and in acetone.

MATTILL, CARMAN and CLAYTON³⁾ demonstrated that the wheatembryo exhausted with ether is ineffective, they found also, that diminution of the fatcontent of a milkpowder diet raised the number of fertile animals.

MILLES and YATES⁴⁾ fed a mixture of casein 15, salts 2, cod liver oil 1, and yellow corn 80. Rats were reproductive on this diet. But if previously the corn was extracted with water they were infertile. Addition of yeast to the latter succeeded in obtaining young in the first generation, but the second one was sterile.

¹⁾ H. M. EVANS & G. O. BURR. The antisterility vitamin fat soluble E. Proc. Nat. Acad. of Science U. S. A. Vol. 11 No. 6 p. 334. 1925.

²⁾ B. SURE. Dietary requirements for reproduction. VI. J. o. Biol. Chem. LXIII. p. 211. 1925.

³⁾ H. A. MATTILL, J. S. CARMAN & M. M. CLAYTON. The effectiveness of the X-substance in preventing sterility in rats on milk rations high in fat. J. o. Biol. Chem. LXI. p. 729. 1925.

⁴⁾ H. G. MILLES & W. W. YATES. The relation of natural foodstuffs and their treatment on growth and reproduction. J. o. Biol. Chem. LXII. p. 259. 1925.

SUPPLEE and DOW¹⁾ investigated the influence of storage of milk-powder on reproduction. They compared relatively fresh powder to such that had been stored for two years. The old powders had been tinned partly without precaution, partly in a CO₂ atmosphere, in a third part an oxygen binding preparation had been used. Casein, saltmixture, agar and dried brewers yeast were added. The rats fed on fresh milk-powder, and those on the stored powder to which an oxygen binding substance had been added, proved fertile, the other lots did not.

A new complication brings the investigations of DANIELS and HUTTON,²⁾ who found, that rats on pure milk diet were mostly infertile, and that the few young born never produced litters. Milkpowder was no better. Addition of ferric citrate gave a little improvement but not much. If 7 grams of sojabeanmeal was added, cooked in 1 l. of milk, reproduction was excellent. Ashes of sojabeanmeal was as well. This in connection with the analysis of the ashes brought them to try mixtures of aluminium-kalium-sulfate, natriumfluoride, natriumsilicate and mangesesulfate which gave very good results till in the sixth generation. There after they tried mixtures of 2 or 3 of the 4 compounds combined with milk; the results however are so surprising, that I doubt if not another factor than the salts may have escaped their attention. A diet of milk and aluminium gives fertility, one with milk aluminium and manganese infertility; milk and silicate give reproduction, milk, silicate and fluoride do not; milk, silicate, fluoride and alumina do.

The authors come to the conclusion that lack of some anorganic compounds in milk causes the failure of reproduction.

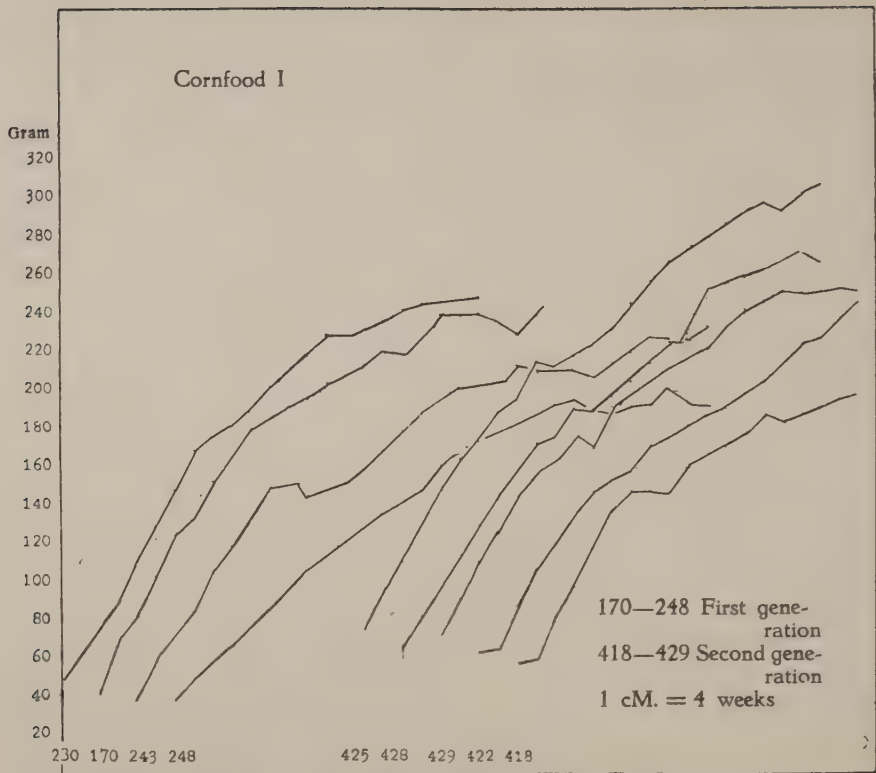
When I consider all the experiments on the relation of milk to reproduction, I come to the conception that milk may contain all the foodstuffs, that are necessary for normal reproduction, but in scarcely sufficient quantity. Now it has been shown, that especially in regard to the accessory foodstuffs the composition of the milk greatly depends on the food of the cow. Moreover the quantity of reserves of foodstuffs the young animals possess in the beginning of the experience varies with the nutrition of their mothers. So it is easily conceivable, how in the case of some experimentators milk proved to be adequate for maintenance of fertility, in that of others not.

Milk has often given rise to discrepancy. HOPKINS and other English workers found 2 cc. milk a day sufficient to complete a diet of artificial food, while American authors wanted 4 to 6 cc. MC. COLLUM was at variance with English investigators on the C-vitamin content of milk. So it will be better to experiment with more uniform compounds.

To prevent confusion it will be necessary too, to discriminate in all our examinations between fertility of males and of females, for it is not likely

¹⁾ G. C. SUPPLEE & O. D. DOW. Reproductive potency of dry milk as affected by oxidation. J. o. Biol. Chem. LXIII. p. 103. 1925.

²⁾ A. L. DANIELS & M. K. HUTTON. Mineral deficiencies of milk as shown by growth and fertility of white rats. J. o. Biol. Chem. LXIII p. 143. 1925.

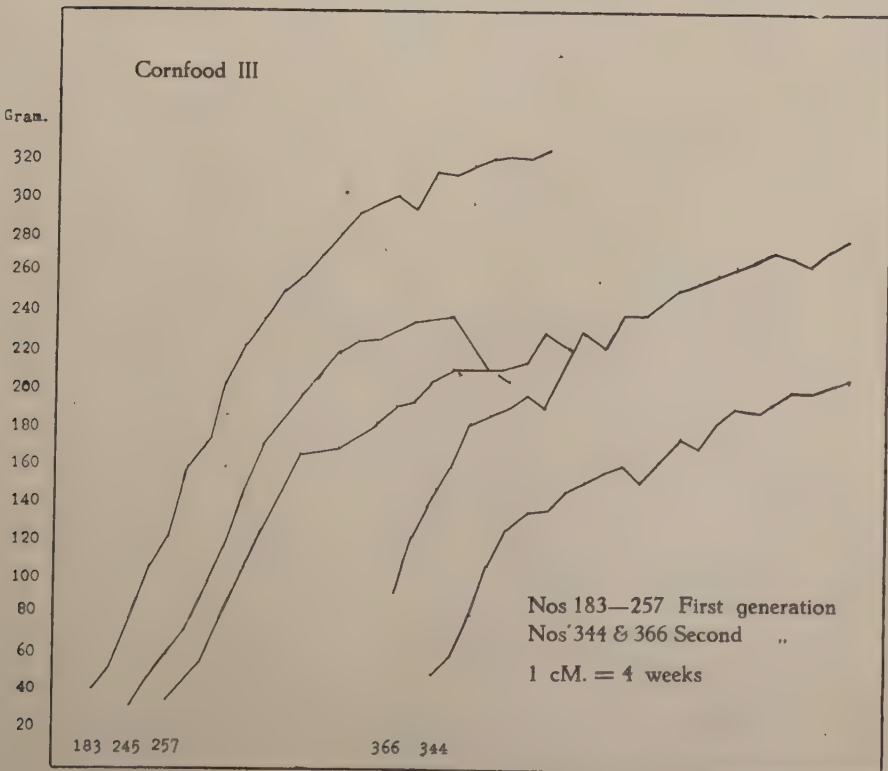
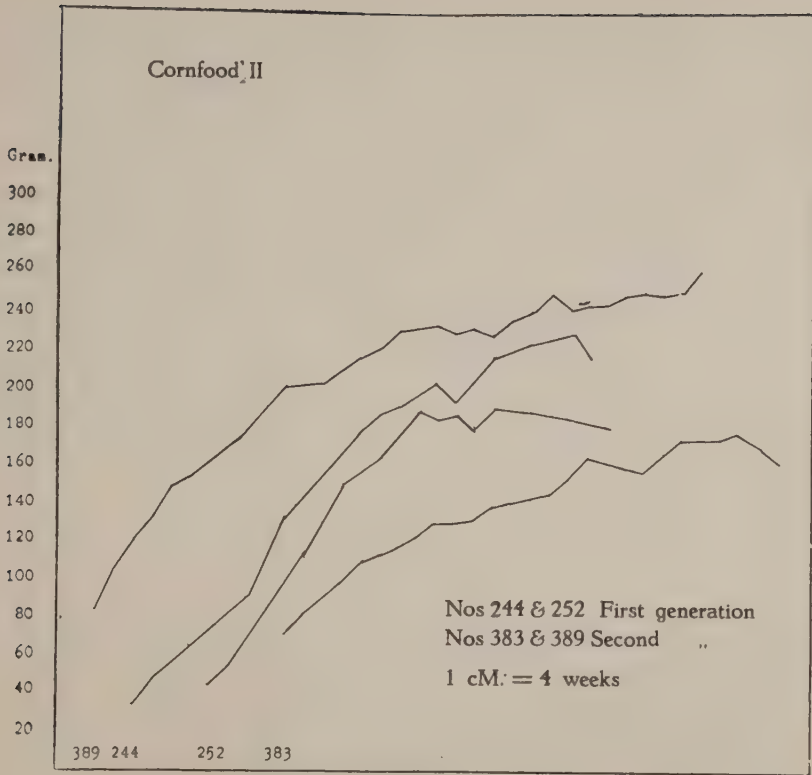


a priori, that in both sexes fertility will depend on the same factor. Then in every case it is to be examined which link in the chain of procreative processes is wrong and whether change of nutrition is able to restore normal reproduction.

In my own experiments on the occurrence of vitamins in different foods it struck me, that often rats did not propagate, although both sexes were confined in a same cage. So I came on studying fertility in relation to diet.

I fed rats on corn, to which cocoanut meal, peanut meal or meat meal had been added, with a little hardened fat, and a saltmixture composed of NaCl 35, KHPO_4 , 7 FeSO_4 3.5, CaCO_3 56, so that all three diets had fairly well the same content of albumen, fat and carbohydrate.

	Cornfood I	Cornfood II	Cornfood III
Corn (Maize)	500	500	500
Peanut meal	100		
Cocoanut meal		200	100
Meat meal			50
Hardenet fat	28	15	22.5
Saltmixture	7.5	8	7



Our normal food is : whole wheat meal 630, meat meal 20, suet 50, salt-mixture 13 and milk 500.

Young rats as soon as they were weaned came on one of the above diets and when about 4 months old were mated. The youngs kept the same food as their mothers. After weaning, sexes were separated till adulthood. The first generation grew well, those on diet I a bit slower than the others, as can be seen in the curves. The second generation also grew well.

Tables 1 and 2 give the results of matings for the first lots. Table 3 and 4 for the II and III. Table 5 gives a general survey of the three experiments.

The young, so far as they did not die, grew as well as their parents ; but matings between them were all infertile but one. Mating from the males of the experiment with females on normal food had no success. Mating of females on cornfood with males on normal food proved fertile, and they raised their youngs.

TABLE 1.
Cornfood I.
1st Generation.

Male N ^o .	Female N ^o .	Number of Young	Weaned
174	172	6	0
	172	0	0
	175	0	0
230	207	0	0
	219	10	6
243	185	11	10
	210	0	0
	260	2	2
	295	6	4
248	209	5	0
	209	0	0
	210	5	0
	210	6	2
	250	5	0
	250	5	3
	254	8	3
	260	6	5

Total 17 matings of which 5 infertile, number of young 75, whereof weaned 35.

We may take from these experiments, that the limiting factors controlling reproduction are others than those controlling growth, and that the demands for nutrition of the males are others than those of the females. I am not

TABLE 2.
Cornfood I with 1 0/0 cod liver oil.
2nd Generation.

Male N ^o .	Female N ^o .	Mated	Result
418	342	17/4 — 23/6	nihil
	394	" "	"
422	430	21/1 — 17/4	"
	426	" "	"
425	417	21/1 — 17/4	"
	421	" "	"
	398	17/4 — 23/6	"
	399	" "	"
427	431	17/4 — 25/5	"
	433	" "	"
428	415	21/1 — 17/4	"
	424	" "	"
	410	17/4 — 23/6	"
	411	" "	"
429	414	17/4 — 23/6	"
	434	" "	"

N.B. The females: 398, 399, 394, 410, 431, 433 and 434 got normal food

Females of second generation mated with males on normal food.

Female N ^o .	Male N ^o .	Mated	Number of young	Weaned
415	436	17/4 — 20/5	8	8
417	400	19/5 — 11/6	4	0
421	400	" "	6	4
424	436	17/4 — 18/5	7	5
426	340	17/4 — 28/5	5	5
430	340	17/4 — 26/6	0	0

6 matings; 1 infertile; 30 young; 22 weaned.

quite sure, that these differences are qualitative and not quantitative ones, though the former seems more likely.

The fact that cod liver oil does not promote reproduction excludes the antirachitic and perhaps the Avitamin, but I have some observations that

TABLE 3.
Cornfood II.
1st Generation.

Male N ^o .	Female N ^o .	Mated	Number of young	Weaned
244	253	24/6 — 15/7	6	5
	258	24/6 — 15/7	8	8
252	242	24/6 — 15/7	2	0
	246	24/6 — 15/7	6	6
	242	22/7 — 15/8	2	2

2nd Generation.

Male N ^o .	Female N ^o .	Mated	Number of young	Weaned
383	375	6/1 — 16/4	0	0
	388	6/1 — 16/4	0	0
	402	16/4 — 10/6	0	0
	405	16/4 — 10/6	0	0
389	373	6/1 — 4/2	1	0
	384	6/1 — 28/1	5	4
	373	16/4 — 8/5	5	5
	384	16/4 — 2/7	7	0

N.B. 402 and 405 got normal food.

Females on Cornfood II mated with males on normal food.

Female N ^o .	Male N ^o .	Mated	Number of young	Weaned
375	438	16/4 — 2/5	4	1
388	438	16/4 — 20/5	1	0

tend to show, that cod liver oil can not fully substitute butter in promoting growth and as the reactions of the E-vitamine, supposed by SURE and EVANS to be the reproduction controlling factor, are very similar to those of A-vitamin, prudence seems me still required.

TABLE 4.
Cornfood III.
1st Generation.

Male N ^o .	Female N ^o .	Mated	Number of young	Weaned
183	195	7/10 — 12/11	7	4
245	255	24/6 — 15/7	8	8
	256	24/6 — 29/7	9	8
257	247	24/6 — 15/7	7	8
	249	24/6 — 15/7	9	9

N.B. Female 182 mated with male on normal food N^o. 85 had 7 young, all weaned.

2nd Generation.

Male N ^o .	Female N ^o .	Mated	Number of young	Weaned
344	363	6/1 — 16/4	0	
	371	6/1 — 16/4	0	
	392	16/4 — 3/5	0	
	397	16/4 — 3/5	0	
366	348	6/1 — 16/4	0	
	356	6/1 — 16/4	0	
	395	16/4 — 25/6	0	
	401	16/4 — 25/6	0	

Females 392, 395, 397 and 401 got normal food.

Females on Cornfood III mated with males on normal food.

Female N ^o .	Male N ^o .	Mated	Number of young	Weaned
348	432	17/4 — 11/5	8	7
356	432	17/4 — 20/5	5	4
371	403	16/4 — 3/6	2	0

TABLE 5.

	Number of males	Number of females	Number of matings	Of which interfile	Number of young	Of which weaned	Young pro mating	Percent weaned
Normal food.....	17	29	52	4	365	319	7	87
Cornfood I:								
1st generation	5	11	17	5	75	35	4.4	46 $\frac{1}{2}$
2nd generation A.....	6	7	10	10	0	0	0	
" B.....	6	6	6	6	0	0	0	
" C.....	5 ¹⁾	7	7	2	30	22	4.3	71
Cornfood II:								
1st generation	2	4	5	0	24	21	4.8	87
2nd generation A.....	2	4	6	2	17	8	2.8	47
" B.....	2	2	2	0	4	1	2	25
Cornfood III:								
1st generation	3	5	5	0	40	34	8	85
2nd generation A.....	2	4	4	4	0	0	0	
" B.....	2	4	4	4	0	0	0	
" C.....	2	3	3	0	15	11	5	73

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Wageningen.*

¹⁾ Big figures mean rats on normal diet.

Physics. — "*On the difference of the fluorescence and the absorption spectra of the uranyl salts.*" By G. H. DIEKE and A. C. S. VAN HEEL. (Communicated by Prof. H. KAMERLINGH ONNES.)

(Communicated at the meeting of June 27, 1925).

1. In 1922 Prof. EHRENFEST has pointed out the necessity to explain a sudden change appearing in the regularly formed spectra of the uranyl compounds, before one could think of solving the structure of these spectra. As is well known nearly all uranyl salts do fluoresce, and with a spectrum of very regular structure. This consists of three to eight groups having the same internal structure and appearing in the spectrum with a constant difference of frequencies. The absorption spectra of the same substances show quite an analogous structure, with the difference however that the frequency interval between the successive groups here has another, always smaller value. For one distinct salt this frequency interval is for example 830 per cm. for the fluorescence and 710 for the absorption.

For different salts these numbers can have a slightly different value, but they do not differ more than 5 per cent. This sudden difference in passing from fluorescence to absorption has to be solved. In the following it will turn out that such a difference must appear necessarily, when we do assume that the bands are to be ascribed to a simultaneous change of the electronic movement and of the oscillation of the nuclei, and that the oscillation energy for the two states of the electron are somewhat different.

2. As the uranium compounds not containing the UO_2 as radical neither fluoresce nor possess such regularly formed absorption spectra as those which do contain it, it seems to us not improbable that we may assume that the energy of the oscillations within this radical does play part at the emission and absorption of light.

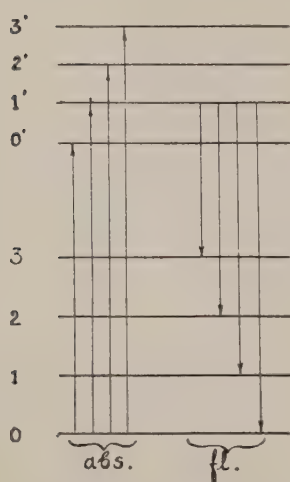
This oscillation energy will generally have a different value before and after the electronic jump, as is the case with the molecules emitting the ordinary band spectra.

In this manner we do arrive at the following scheme of niveau's.

For a non-excited molecule : a state 0 without oscillation energy ; and states 1, 2, 3, etc. with respectively once, twice, thrice, etc. the energy of the fundamental oscillation.

For an excited molecule : a state 0' again without oscillation energy ;

and states $1'$, $2'$, $3'$, etc. with respectively once, twice, thrice, etc. the oscillation energy in excited state.



In the figure these niveau's are schematically represented.

At low temperatures there are only molecules on niveau 0. Absorption then will take place from this niveau to the different upper niveau's, *the absorption bands will therefore show a frequency interval equal to the interval between the oscillation niveau's in excited state.* The corresponding lines are given in the figure.

From a definite excited niveau fluorescence will take place by jumping back to the different lower niveau's.

The frequency interval of the fluorescence bands will therefore correspond to the change

of oscillation energy in non-excited state.

This should explain the plunge of the constant frequency interval in passing from fluorescence to absorption.

For convenience' sake we have assumed in the foregoing that in the states 0 and $0'$ no oscillation energy is present. Should this be the case then we ought to speak of *changes* of oscillation energy. One must confer to these changes the quantum numbers 1, 2, 3, etc., resp. $1'$, $2'$, $3'$, etc. in order to obtain the niveau's corresponding to these numbers.

This simple scheme does not suffice to explain the complete fluorescence and absorption spectrum of an uranyl salt. It has appeared to us however that a little extension suffices. We hope to publish before long about this subject.

In conclusion we will point out that the fact, that the frequency difference is constant in such a high degree, indicates that the oscillations are to a high degree harmonical.

Physics. — "*On the monochromatic excitation of fluorescence.* By A. C. S. VAN HEEL. (Communication from the Physical Laboratory at Leyden.) (Communicated by Prof. H. KAMERLINGH ONNES.)

(Communicated at the meeting of June 27, 1925).

It is a good many years ago that the idea arose to continue the investigation on the fluorescence of the uranyl salts at low temperatures done by H. and J. BECQUEREL and H. KAMERLINGH ONNES in 1909 at Leyden. The war and the succeeding unfavourable circumstances did prevent for many years the execution of this project. Only in 1923 it came to the preparation of the intended measurements. From the first the old project was prominent to investigate the relation between the intensity of the fluorescence bands (better still of the components of these bands, that can indeed be separated sufficiently at hydrogen temperatures) and the wave length of the exciting band.

The importance of this investigation, which is from the purely experimental point of view highly interesting enough, is very increased by the considerations given by Prof. EHRENFEST in the "*Gedenkboek H. KAMERLINGH ONNES, 1922*".

Prof. KAMERLINGH ONNES had the kindness to invite me to help at the preliminary observations, while the definitive experiments should take place with bigger and better instruments on a larger scale and with greater precision. Meanwhile it appeared that with the provisional contrivement there could be obtained important results. Prof. BECQUEREL and Prof. KAMERLINGH ONNES were then willing enough with kind approval of Prof. DE HAAS to leave the subject to me. Here follows a preliminary communication of the results obtained up to now.

Concerning the work done in the range of years after 1909 we have to mention, that E. L. NICHOLS and his co-workers have published in the *Physical Review* a series of articles, containing a description of the fluorescence and absorption of uranyl salts at temperatures between 20° and -185° C. The regularities found by H. BECQUEREL have been found by them in all investigated cases.

The fluorescence consists of (at the utmost) eight bands, each of which consists of pl.m. five components, which components are separated at low temperatures and increased in number with many less regularly appearing bands. The interval in frequency scale between homologous components is constant over the whole of the fluorescence spectrum.

The absorption spectrum of these compounds too shows a similar regular structure. This too NICHOLS and his co-workers found confirmed. In contra-distinction however to what H. BECQUEREL formerly found.

they got the conviction that the interval appearing in the absorption spectra has another, always smaller value than the interval in the fluorescence spectra.

When one considers the spectra of one substance, one gets the impression, that every series of fluorescence lines is prolonged towards the violet in a corresponding absorption series, while the constant frequency interval plunges from $\Delta \frac{1}{\lambda} = \text{pl.m. } 82$ (λ expressed in mm.) to $\Delta \frac{1}{\lambda} = \text{pl.m. } 71$.

For this investigation I was allowed to dispose of the same crystal of autunite, that has served at the former experiments of Prof. BECQUEREL and also at those in 1909. The choice fell on this substance (double phosphate of uranyl and calcium) on account of the small number of components in every group of fluorescence and absorption, which renders superfluous the great monochromation necessary for the other substances, that do fluoresce with more lines. Low temperatures (liquid air) is wanted for the separation of the groups. In liquid hydrogen the phenomenon will be more sharply defined.

As a monochromator was used the instrument, described by Prof. ZEEMAN in Arch. Néerl., T. 27, p. 252. The aperture of the lenses is $f/2.2$. We were allowed to use a glass spectrograph of BUSCH (aperture $f/3$).

Results :

First there has been established, that a considerable fluorescence never can be excited by light that has not precisely the wavelength of one of the absorption lines. The extreme fluorescence band on the violet side is, as is known, at the same time the extreme absorption band on the red side. I call here this band 1, and number the fluorescence bands towards the red with 2, 3, etc. and the absorption bands towards the violet with 2', 3', etc. Band 1 might thus be called 1', too.

Now when the substance is illuminated by light of the wavelength of fluorescence band 2, then there is no fluorescence observable at all in the spectrum. When the crystal is illuminated by light of the wavelength of the reversible band 1, then there is fluorescence with the two fluorescence bands 2 and 3. It is not settled whether the reversible band emits in this case.

When one excites with the still more violet band 2', then the fluorescence of the bands 2 and 3 augments, the reversible band begins to emit and a still more red fluorescence band (4) appears.

Quite analogous phenomena are shown by the double sulphate of potassium and uranyl.

These observations have been recorded photographically at the temperature of liquid air. The observations in liquid hydrogen were done visually.

Finally I want to express my thanks for the use of the instruments and also of the kindness of the said professors.



Fig. 1. Fluorescence spectra of the double sulphate of potassium and uranyl at -185°C . Red to the left, violet to the right.

- a. The whole of the fluorescence spectrum, as excited by a broad range in the ultra-violet.
- b. Fl.-spectrum, excited by one ultra-violet absorption group (the reversible band 1 is not visible on the reproduction).
- c. Fl.-spectrum, excited by absorption group 2' (here too the yet present fluorescence group 1 is not visible on the reproduction).
- d. Fl.-spectrum, weakly appearing, as the exciting light lies between absorption groups.
- e. Fl.-spectrum, excited by the reversible group 1.
- f. Absence of fluorescence when the exciting light has the wavelenght of the fluorescence group 2.

Remark: at all excitements not only the strong, but also the weaker components of the fluorescence groups appeared.

Physics. — "*Measurements of the surface tension of liquid helium*".

Communication N^o. 179a from the Physical Laboratory, Leyden.

By A. TH. VAN URK, W. H. KEESOM and H. KAMERLINGH ONNES.

(Communicated at the meeting of October 31, 1925).

§ 1. *Method and apparatus.* In order to determine the surface tension of liquid helium in contact with its saturated vapour we used, as did KAMERLINGH ONNES and KUYPERS ¹⁾ in the case of hydrogen, the method



Fig. 1.

of the capillary elevation in a narrow tube. To diminish irregular pressure differences at the open ends of the capillaries these were made comparatively short (Fig. 1, c.f. Fig. 1, Comm. N^o. 142d). Also it appeared desirable to have an immediate control as to whether or not the measured rise was the true rise corresponding to the temperature used. It often happened that gas bubbles rose in the narrow tube, causing the meniscus to oscillate for a considerable time and then to remain stationary for some minutes at an entirely wrong height. By placing two capillaries of about the same diameter alongside one another in the same wide tube and by comparing the elevations in both, this uncertainty was eliminated. It is hardly to be expected that precisely the same disturbance to the elevation would appear in both tubes at the same time.

A platinum wire bearing two loops, in which the capillaries were fixed with a little Kothinsky cement, was fused into the wall of the wide tube.

§ 2. *Calibrations.* Two capillaries about 10 cms. long were chosen, one twice as wide as the other. Each capillary was calibrated several times by two different methods, namely, by filling with mercury threads of various lengths which were measured, the mercury used being weighed, and by placing the tube filled with mercury in a liquid (an approximately 10:1 mixture of chloralhydrate and glycerine) of the same refractive index, for Na-light, as glass and then measuring the diameter of the mercury thread with the micrometer eyepiece of a microscope.

The two calibrations of both tubes (of radius about 0,009 and 0,02 cms.) could not be made to agree to more than $\frac{1}{2}\%$. A calibration

¹⁾ Comm. N^o. 142d. These Proceedings 17, 525, 1914.

curve was drawn for each capillary from which the radius at the position of the meniscus could be read off. From each tube two pieces about 3 cms. long were taken to be placed alongside one another.

§ 3. *Observations.* The positions of the menisci in the capillary tubes (the top and the edge could not be distinguished) and the positions of the liquid in the wide tube, in the neighbourhood of the capillaries and somewhat further off, were measured. Besides these the tops of the tubes were also read to determine the positions of the menisci in relation to the tubes.

The corrected rise ' H ' was calculated from the rise ' h ' measured in a capillary tube placed inside a wider tube by introducing the following two corrections:

$$H = h + \frac{r}{3} + h'$$

where r is the inner radius of the capillary and

$$h' = \left(h + \frac{r}{3} \right) \frac{\frac{2d}{(R-r_1)^2}}{1 - \frac{2d}{r(R-r_1)^2}} \quad ^1)$$

¹⁾ See J. E. VERSCHAFFELT, Leyden Comm. N^o. 18. These Proc. 1895. This correction is not properly applied in Communication N^o. 142d. For the there communicated measurements of *Hydrogen* we now can borrow the above corrections from the elevation in two concentric tubes as these have been more accurately calculated by a graphical method by VERSCHAFFELT, Bull. Acad. Roy. de Belg. Cl. d. Sc. 1921, p. 574. The following values for H , which are to be substituted for the values given in Table I of Communication N^o. 142d were obtained.

The corresponding values of ψ_σ are also given (c. f. Table II, Comm. N^o. 142d)

T	H	ψ_σ
20.40	1.694	1.910
18.70	1.883	2.194 ⁵
17.99	1.962	2.319 ⁵
16.16	2.169	2.631
14.68	2.324	2.860

(c. f. also J. E. VERSCHAFFELT, Comm. Phys. Lab. Ghent, N^o. 2, Wis- en Natuurkundig Tijdschrift 2, 231, 1925). In the formula of VAN DER WAALS $\psi_\sigma = A(1-t)^B$ the values of A and B become $A = 5.554$, $B = 1.119$, and the constant of EÖTVÖS $k_{E\ddot{o}} = 1.363$, and EINSTEIN's constant, 6.64×10^{-9} .

In the case of *helium* we have been unable to use the new calculations of the rise between two concentric tubes by VERSCHAFFELT, as the capillary constant of helium is small and the table given by him does not go far enough. However the difference lies within the experimental errors.

The combination of both corrections gives

$$H = \frac{h + \frac{r}{3}}{1 - \frac{2dr}{(R-r_1)^2}}.$$

We have applied this formula to the present case in which two tubes are placed in a wider tube; as the total correction h' amounts to 1 % for the wider capillary, and only $1/2$ % for the narrower, any error thus involved is negligible.

Corrections for the expansion of the glass have been neglected as they would certainly lie within the experimental error.

The temperature was calculated from the vapour pressure according to the formula given in Comm. N^o. 147b ¹⁾.

§ 4. *Results.* The differences between the two values found for Hr at the same temperature amount to a maximum of 5 %. It must be remembered that the largest rise was 5 mm. and that this had to be read through two vacuum vessels and then through the walls of two glass tubes.

The surface tension ψ_σ and the molecular surface tension ψ_M (in ergs) can be calculated from the product Hr by means of the following formulae ²⁾

$$\psi_\sigma = (\rho_l - \rho_v) \frac{g}{2} \cdot Hr.$$

$$\psi_M = \psi_\sigma \left(\frac{M}{\rho_l} \right)^{2/3}.$$

The results are shown in Figure 2. The points \odot and Δ refer to the measurements with the capillaries of radius 0.02 and 0.009 cms. respectively. The values of ψ_M given in table II have been interpolated by means of this graph. To them correspond the values given for $a^2 = Hr$, a being the capillary constant.

TABLE I.

T	ψ_M	a^2	T	ψ_M	a^2
4.20	0.98	0.00181	2.50	2.69	0.00419
4.00	1.19	211	2.00	3.08	477
3.50	1.68	280	1.50	3.22	496
3.00	2.19	350			

¹⁾ These Proceedings 18, 493, 1915.

²⁾ The densities ρ_l and ρ_v for helium were obtained from E. MATHIAS, C. A. CROMMELIN, H. KAMERLINGH ONNES and J. C. SWALLOW, these Proceedings 28, 526, 1925, Leyden Comm. N^o. 172b.

§ 5. Discussion.

From Fig. 2 it is seen that the line which represents the surface tension as a function of temperature has a curvature which first appears at 2.4° K. Above this temperature the line appears to be quite straight.

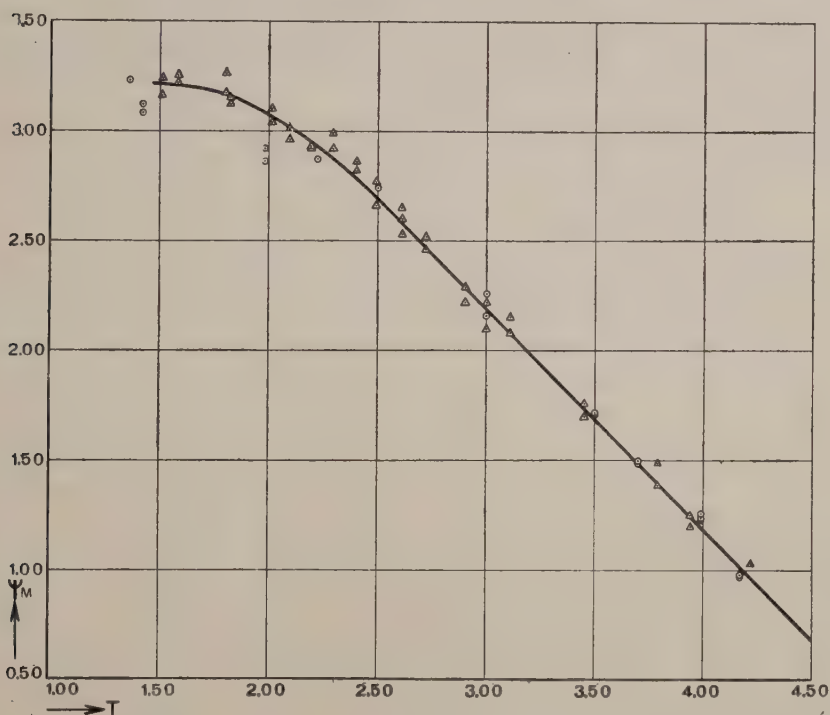


Fig. 2.

We have calculated the EÖTVÖS' constant¹⁾ for the straight portion and this amounts to about 1.0. Thus helium obeys the rule, stated by KAMERLINGH ONNES and KEESOM²⁾, according to which normal substances in this respect form a series in which the EÖTVÖS' constant increases with the critical temperature. According to this rule the constant for helium must be the smallest, which is actually the case.

The curvature just mentioned probably indicates, as does the maximum that is observed in the liquid density³⁾, a peculiarity in the molecular

¹⁾ $d\psi_M/dT$. See „die Zustandsgleichung“. *Enz. d. math. Wiss.* V. 10, Leyden Suppl. N^o. 23, § 37b.

²⁾ „Die Zustandsgleichung“, note 381.

³⁾ The observations of this phenomenon published up to now only indicate the existence of a maximum of the apparent density of liquid helium in a glass vessel. Here it can be further stated that measurements have been made in which a graduated silver scale was placed next to the glass scale of the reservoir in the helium bath and both scales compared with the aid of a cathetometer. No difference in the expansions of either scale could be recorded, from which it can be concluded that such a difference, if it exists, amounts to less than a thousandth part of the calculated density of the liquid helium. It is hardly

attraction. With regard to the former no such curvature is observed with the other so-called permanent gases. The surface tension of the latter, however, because of the appearance of the solid state, cannot be observed at such a low reduced temperature as is the case with helium.

We wish to acknowledge our very great indebtedness to Mr. G. P. NIJHOFF, who made part of the observations and to Miss A. F. J. JANSEN and Mr. J. VOOGD for their help in the temperature measurements.

possible that both substances undergo the same transformation at the same temperature; hence it follows that the maximum of the apparent density of the helium in a glass reservoir is due to the helium itself.

Physics. — *"Isotherms of di-atomic substances and their binary mixtures. XXXIII. Isotherms of oxygen between -40° C. and $-152^{\circ}.5$ C. and pressures from 3 to 9 atmospheres. (Communication N^o. 179b from the Physical Laboratory of the University of Leyden.)* By G. P. NIJHOFF and W. H. KEESOM.

(Communicated at the meeting of October 31, 1925).

1. Introduction.

A knowledge as accurate as possible of the first virial-coefficients, i.e. of B , afterwards of C , is important, as well in view of the calculation of gas-densities in the neighbourhood of the normal one, and the determination of corrections of the gas-thermometer, as in view of the fact, that in practice it is yet possible to infer for those coefficients some relations with molecular structure and forces, eventually the molecular volume and the moment of the molecular quadruplet or doublet, using simplifying assumptions that are fairly approximate.

For the determination of B are to be preferred measurements with gases in slightly compressed states, in which the influence of B is already important and that of C might be noticeable, but the following virial-coefficients have no influence. Since at very low temperatures, in the first place with substances with a low critical pressure, such states are reached at about atmospheric pressures, and on the other hand, at high pressures and with substances with higher critical pressures those states extend to pressures as high as some tens of atmospheres, for some gases those states exist at a given temperature-range with pressures of only some atmospheres. So we desired to complete the investigations of isotherms at low temperatures, done in Leyden at pressures from about 20 to 100 atmospheres, with measurements at pressures from 3 to 20 atmospheres, to fill up a gap between those first-mentioned and gas-thermometer measurements at the lower pressures.

In the second place precise measurements at those pressures are of importance with substances with a very low critical pressure, e.g. helium, because then one gets the same reduced pressures at which other gases (at higher pressures) have been studied, and so a comparison after the principle of the corresponding states will be possible.

To have some estimation of the precision that is to be reached in the pressure-domain mentioned, we began with measurements with oxygen, for which substance KAMERLINGH ONNES and KUYPERS¹⁾ determined the isotherms with pressures from 20 to 60 atmospheres and temperatures from

¹⁾ H. A. KUYPERS and H. KAMERLINGH ONNES, Arch. Néerl. sér. III A, 6, 277, 1923. Leyden. Comm. N^o. 165a, H. KAMERLINGH ONNES and H. A. KUYPERS. Leyden Comm. N^o. 169a.

20° C. to —117° C. We were able to extend the measurements down to —152° 5 C., before approaching condensation made us stop.

2. *The apparatus.*

a. The oxygen was prepared with the same apparatus as described in Leyden Comm. N^o. 165a ¹⁾).

b. *The piezometer.* For the arrangement we refer to Leyden Comm. N^o. 69 ²⁾). We calibrated a stem 108 cm. long, with a total volume of 108.6 ccm. A gas-reservoir of 137.8 ccm. capacity was blown on this stem. Thus it is possible to vary the densities in the piezometer reservoir rather widely. The same result could be reached by using instead of such a big stem a smaller piezometer reservoir; but then the accuracy would be less.

c. *The piezometer reservoir* had a capacity of 20.400 ccm., and was connected in the usual way by means of a steel capillary to the stem.

d. *The pressure measurements.* Pressures for these measurements varied from 9 to 3 atmospheres. They were measured first by means of the standard open manometer of KAMERLINGH ONNES ³⁾). In this manometer columns of mercury of a constant length of 3 meters are coupled in series by air transmission, and a first tube of variable length is added to fill continuously the 4 atm. intervals. All tubes are read with telescopes, and tenths of mm. estimated. To know the pressure of the gas in the piezometer one has to add the barometer pressure to the so measured pressure and to subtract from it the pressure of the mercury column from the top of the meniscus in the piezometer to that in the indicator glass ⁴⁾). With low pressures the difficulty arises, that the uncertainty (which is about 0.3 mm.) in reading the variable first tube, especially in cases where the mercury column happens to be very short, can give a relatively large error, which is not now compensated by the much larger accuracy (of 1/10.000) of the other tubes, as it is with greater pressures.

Also the inaccuracy in the reading of the mercury column in the piezometer, (where the height in the indicator-glass is estimated in $\frac{1}{10}$ mm.) was not to be neglected against the small pressures, which were now to be measured. Therefore the indicator glass was read with a small cathetometer, whereas the open tube of the open standard manometer was replaced by a small divided manometer, constructed after the model of the standard manometer, and joined parallel to the open tube. This small manometer could be read with a cathetometer. For greater simplicity the first tube of this small manometer was closed, to eliminate the reading of the barometer. The apparatus was constructed in the following way.

¹⁾ Arch. Néerl. sér. III A, 6, 277, 1923, Leyden Comm. No. 169 a.

²⁾ These Proceedings 3, 621, 1901.

³⁾ These Proceedings 1, 213, 1898, Leyden Comm. N^o. 44.

⁴⁾ The indicator-glass is given at C₃ in Plate I, these Proceedings 3, 621, 1901, Leyden, Comm. No. 69.

e. *The divided manometer for pressures from 1 to 4 atm.* Two tubes of mercury of a constant length of 115 cm. can be coupled in series and a third tube is affixed by means of which the pressure can be changed continuously. These tubes are connected, in the same manner as the open manometer, to a pressure tube, to which are also connected: a cock (K_1), to apply the pressure, a cock (K_2), to shut off the connection with the open manometer, and two cocks (K_3 and K_4), the first an ordinary outlet, the second connected with a small vacuum pump, for reducing the external pressure on the closed tube.

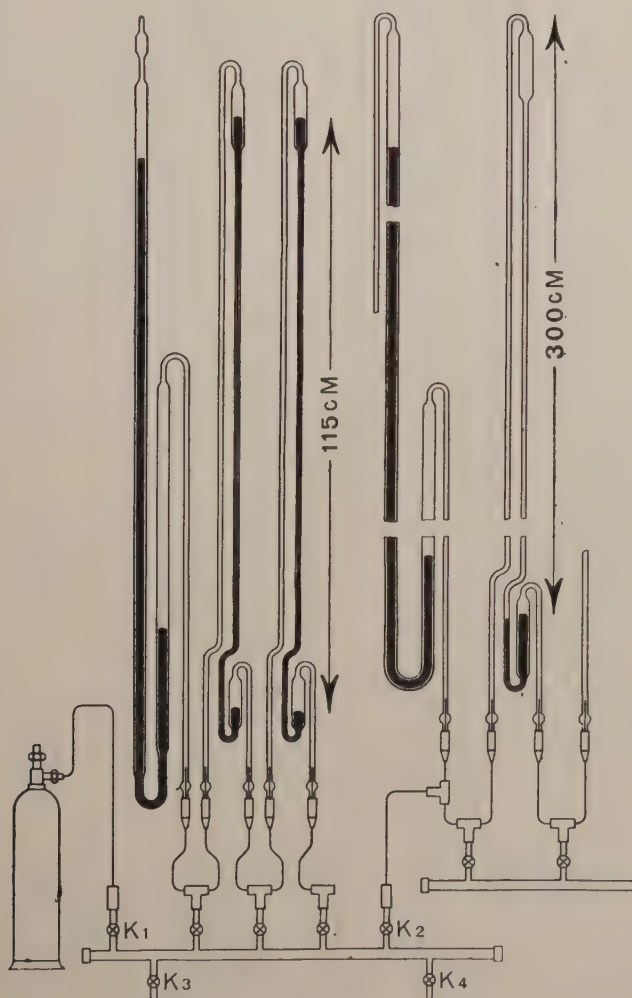


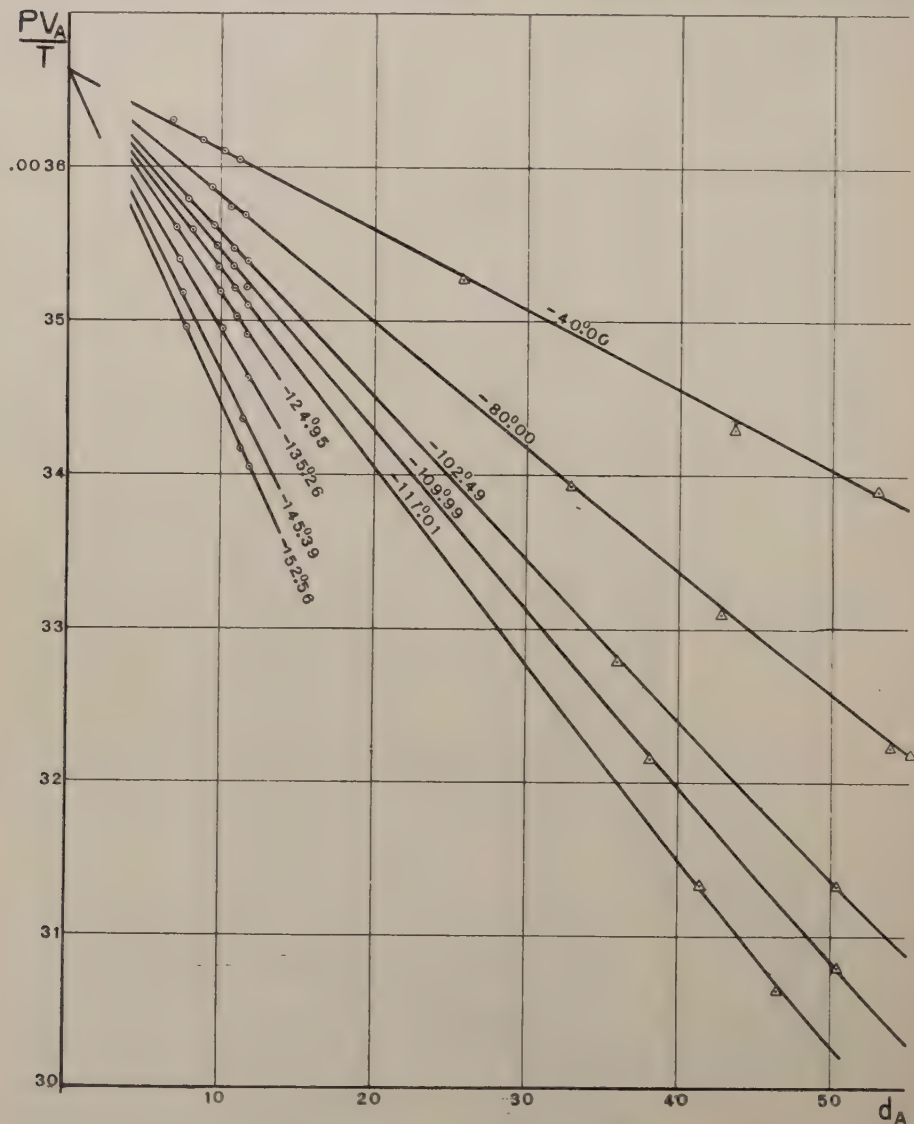
Fig. 1.

When measuring, one first sets the correct pressure on the open manometer, keeping the connection with the small manometer closed. From the length of the mercury column in the open tube of the standard manometer

we can calculate how much pressure to apply to the small manometer. The pressure in the latter can always be measured to an accuracy of one in 10.000, because the measurements are made with a cathetometer.

3. The measurements.

a. The *normal volume* was determined before and after the measurements in the way described in Leyden Comm. N^o. 78 ¹⁾. Because the volume of the upper part of the stem reaching outside the waterbath would be



△ KAMERLINGH ONNES and KUYPERS.

● NIJHOFF and KEESOM.

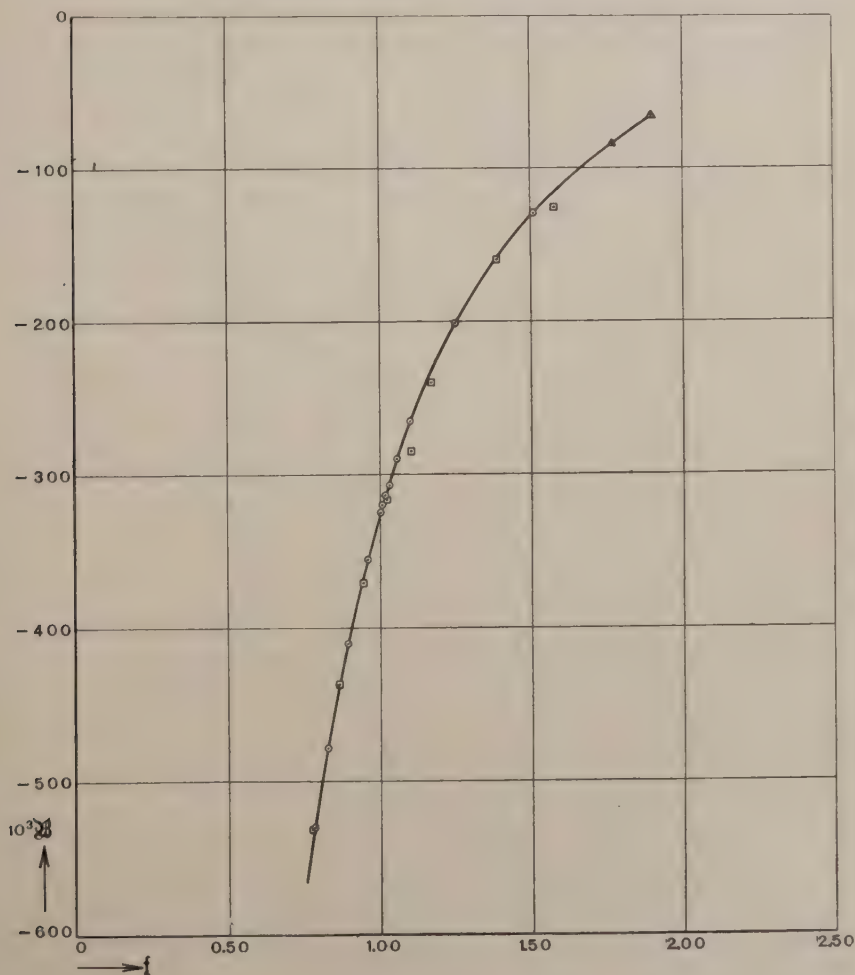
Fig. 2.

¹⁾ These Proceedings 4, 761, 1902.

disturbing for such a small gas reservoir, the whole stem was put in a separate waterbath, which was fed by a sidebranch from the same thermostat which kept the main waterbath at constant temperature. The temperature of the two baths agreed within $1/100$ of a degree.

b. The temperatures were obtained with a cryostat described in Leyden Comm. No. 83 ¹⁾, using methyl-chloride and ethylene.

4. The results of the measurements are given in the following tables.



△ KUYPERS and KAMERLINGH ONNES.

□ CATH and KAMERLINGH ONNES.

○ NIJHOFF and KEESOM.

Fig. 3.

¹⁾ These Proceedings 5, 502, 1903.

TABLE I.

Normal volumes before: 254.37

after: 254.32

θ	p	d_A	p^v_A	$\frac{p^v_A}{T}$	$p^v_A \text{ obs.} - p^v_A \text{ calc.}$
— 40.01	9.382	11.165	0.8403	0.003605	+ 0.00005
	8.559	10.169	8416	3611	+ 1
	7.385	8.756	8432	3618	0
	5.755	6.801	8462	3631	+ 5
— 80.00	7.918	11.490	6892	3569	+ 1
	7.300	10.576	6902	3574	— 4
	6.445	9.306	6926	3587	— 1
—102.49	7.029	11.650	6037	3539	+ 4
	6.507	10.752	6051	3547	+ 1
	5.783	9.519	6076	3562	+ 2
	4.759	7.795	6105	3579	— 0 ⁵
—109.99	6.684	11.634	5745	3522	+ 5
	6.218	10.781	5767	3536	+ 1 ⁵
	5.594	9.666	5788	3549	+ 0 ⁵
—113.94	6.501	11.616	5597	3517	0
	6.092	10.855	5613	3527	+ 1
	[5.470	9.695	5642	3545	+ 6]
—116.01	6.486	11.767	5512	3509	— 4
	6.008	10.855	5534	3523	0
	5.422	9.757	5557	3538	0
	4.506	8.055	5594	3561	+ 3
—116.03	4.533	8.101	5596	3563	+ 1
	3.841	6.839	5616	3576	— 1
—117.01	6.389	11.661	5478	3510	0
	5.965	10.854	5496	3521	+ 1
	5.385	9.760	5517	3555	— 1
	4.493	8.088	5555	3559	+ 2

TABLE I (Continued).

θ	d	d_A	p^v_A	$\frac{p^v_A}{T}$	$p^v_A \text{ obs.} - p^v_A \text{ calc.}$
-118.58	6.254	11.538	0.5421	0.003508	- 0.0001
	5.909 ⁵	10.876	5433	3516	- 3
	3.799	6.883	5520	3572	+ 1
-124.95	6.013	11.629	5171	3491	- 4
	5.688	10.961	5190	3503	+ 0 ⁵
	5.160	9.897	5213	3519	0
	3.715	7.042	5276	3561	+ 1
	2.882	5.422	5316	3588	+ 5
	5.599	11.732	4773	3463	- 1 ⁵
-135.29	4.838	10.045	4816	3495	+ 2
	3.550	7.277	4878	3540	- 1
	5.007	11.411	4387	3436	- 3
-145.39	3.375	7.513	4492	3518	+ 3
	4.854	11.827	4102	3405	- 1
-152.56	4.636	11.262	4117	3417	- 1
	3.251	7.716	4213	3496	0

In the last column the differences are given between the measured p^v_A 's, and those calculated with help of the B_A 's of table II.

The points also are given on a $\frac{p^v_A}{T}$, d_A -graph (fig. 2), where the points measured by KAMERLINGH ONNES and KUYPERS are appended as far as possible. They agree within the limits of accuracy ¹⁾.

In the second table we give a survey of the B -values, as calculated graphically from our measurements. For greater completeness we add those for 20° C. and 0° C. from Leyden Comm. No. 165a. All are given together with those of CATH and KAMERLINGH ONNES, as given by VAN AGT in Leyden Comm. No. 176c ²⁾, in another graph (fig. 3). They too agree within the limits of accuracy.

In conclusion we want to express our sincere thanks to those who helped

¹⁾ For greater clearness two isothermals are omitted in fig. 2, viz. -118° C. and -116° C. (-117° C. is given).

²⁾ These Proceedings 28, 687, 1925.

us by regulating the temperatures, especially Mr. J. VOOGD, who helped us with the measurements and the calculations.

TABEL II.

θ	t	A_A	$10^3 \cdot B_A$	$10^3 B$	\mathfrak{B}
+20° C.	1.8998 ⁵	1.07426	—0.80379	—0.74823	—65.846
0	1.7702	1.000952	0.95803	0.95712	84.229
—40.01	1.5109	0.85430	1.258	1.473	129.59
—80.00	1.2516	70773	1.619	2.288	201.31
—102.49	1.1058 ⁵	62530	1.885	3.014	265.24
—109.99	1.0572	59781	1.967 ⁵	3.291	289.60
—113.94	1.0316	58333	2.038	3.494	307.46
—116.01	1.0182	57574	2.051	3.562	313.50
—117.01	1.0117	57208	2.079	3.634	319.81
—118.58	1.0016	56632	2.090	3.690 ⁵	324.77
—124.95	0.9603	54298	2.192	4.037	355.27
—135.29	0.89324	50508	2.355	4.663	410.32
—145.39	0.82777	46806	2.545	5.437	478.50
—152.56	0.78129	44178	2.660	6.021	529.87

$$T_k = 154.27 \quad p_k = 49.713^1) \quad B = \frac{B_A}{A_A} \quad \mathfrak{B} = \frac{p_k}{RT_k} \cdot B = 88.0024 B.$$

¹⁾ For the critical data see: These Proceedings **13**, 939, 1911. Leyden Comm. No. 117.

Physics. — "*Further experiments with liquid helium. C A. On the properties of supraconductive metals in the form of thin films.*" By G. J. SIZOO and H. KAMERLINGH ONNES. (Comm. N^o. 180a from the Physical Laboratory at Leyden.)

(Communicated at the meeting of October 31, 1925).

§ 1. In the report of KAMERLINGH ONNES to the IVth Conseil-Solvay (April 1924) a preliminary communication was made on the results of an investigation, at that time not yet considered as finished, concerning the behaviour of supraconductive metals in the form of thin films.

Though we intended to continue and to repeat with more accuracy these experiments, the difficulties which arose as well as the necessity of other researches made it impossible for us to work out this plan. However we consider the results obtained, notwithstanding their incompleteness, important enough to be communicated here in their quantitative form.

The original plan to investigate extremely thin films, a few $\mu\mu$ thick for example, had soon to be given up, as such very thin lead or tin films, obtained by evaporation or cathode sputtering, showed the phenomenon of agglomeration or coalescence in such a high degree, that their electrical resistance increased very quickly with the time and soon reached an infinite value.

In the literature on the properties of thin metallic films this phenomenon is repeatedly mentioned ¹⁾.

The impact of the particles in the course of one or more days could be followed easily with the aid of a microscope. Keeping the films at liquid air temperature appeared to decrease the speed of the agglomeration process. Still it remained so inconvenient, that we decided to investigate first only thicker films. All measurements to be communicated here, are therefore made with films of a thickness of 0.3—0.5 μ , obtained by cathode sputtering. Even the resistance of these thicker films appeared to increase with the time, though in a much smaller degree than that of the very thin ones. The thickness could be determined by weighing.

For the resistance measurements it was necessary to apply four contacts to the film. For that purpose the four corners of the glassplate, on which the film had to be deposited were first sputtered with a thick layer of the metal, the rest of the plate being covered with mica.

Four wires of the same metal, with flattened ends, were then fixed to the four corners by means of little copper clamps.

The glassplate provided in this way with contacts and leads, was then

¹⁾ E.g. L. R. KOLLER. Physical properties of thin metallic films. Phys. Rev. XVIII (1921) p. 221.

placed again, in the sputtering jar, to apply on it the film of the required thickness.

The sputtering took place in a glass bell jar which contained a cathode of the metal to be sputtered and which was placed on a copper plate as anode. The jar was filled with hydrogen.

The resistances were always measured by comparing the potentials at the end of the unknown and a known resistance placed in series with each other. This was done with a compensation apparatus, free from thermo-effects, or when this apparatus was in use for other measurements, with a set in which the deflections of a galvanometer were proportional to the mentioned potentials. With the measurements in liquid helium the films were placed in the heliumcryostat shown in Leyden Comm. N^o. 124c (1911), fig. 4¹).

The temperatures were calculated with the aid of the formula of Leyden Comm. N^o. 147b, (1915) p. 33²) from the vapour pressures of the helium-bath, which were read on a closed mercury manometer.

§ 2. The greater part of the measurements are made with tin films. Of one of them the temp. coeff. of the electrical resistance between 0° and 100° C. was determined. It was found to be 0.0034. Of another film the change of the resistance in the region between roomtemperature and helium temperatures was followed. Of course these measurements could not be made in one day. In consequence of this fact connected with the process of the change in resistance, the results can only give a rough survey of the way in which the resistance of the film depends on the temperature. They are sufficient though to show immediately the great difference between the behaviour of these films and that of ordinary tinwires.

The results of these measurements are given in table I. The low

TABLE I.

Date	T	$W_{T_{pl}}$
18—V—1922	roomtemp.	9 Ω
19—V—1922	4°.22 K.	1.96 ..
15—VI—1922	83.04	5.19 ..
15—VI—1922	90.18	5.46 ..
17—VI—1922	114.36	6.51 ..
21—VI—1922	228.55	10.93 ..
22—VI—1922	roomtemp.	13.55 ..

¹) These Proceedings 14, 1911. p. 682.

²) These Proceedings 18, 1915, p. 507.

temperatures were obtained by means of suitable cryostat liquids in the way generally used in the Leyden laboratory and measured with calibrated platinum thermometers.

At heliumtemperatures a number of measurements were carried out with the films Tp I, Tp II and Tp II'. In the first place the temperatures of the "vanishing point" of each film were determined. These appeared to be different for different films and also for the same film (Tp II') on different data.

TABLE II.

Date	p_{helium} in mm Hg	T	$W_{T_{pI}}$	$\frac{W}{W_k}$
18 — V — 1922		roomtemp.	9 Ω	1.
19 — V — 1922	760	4.20 ₀	1.96 ..	0.218
	480	3.76 ₄	1.964 ..	0.2182
	444	3.69 ₈	1.817 ..	0.2019
	424	3.66 ₈	1.518 ..	0.1686
	404	3.62 ₀	0.408 ..	0.0453
	390	3.59 ₂	0.0031 ..	0.00035
	387	3.58 ₆	0.00014 ..	0.00015
	386	3.58 ₄	0.00000 ..	0.00000

TABLE III.

Date	p_{helium} in mm Hg	T	$W_{T_{pII}}$	$\frac{W}{W_k}$
8 — VI — 1922		roomtemp.	32 Ω	1.
9 — VI — 1922	485	3.56 ₀	3.480 ..	0.1088
	450	3.70 ₈	3.486 ..	0.1089
	430	3.67 ₂	3.486 ..	0.1089
	410	3.63 ₂	3.480 ..	0.1088
	390	3.59 ₂	3.510 ..	0.1097
	370	3.55 ₀	3.380 ..	0.1057
	350	3.50 ₈	2.893 ..	0.09041
	330	3.46 ₂	0.304 ..	0.00950
	310	3.41 ₈	0.008 ..	0.00024

TABLE IV.

Date	P_{helium} in mm. Hg	T	$W_{T_{pII'}}$	$\frac{W}{W_k}$
15 — VI — 1922		roomtemp.	110 Ω	1.
16 — VI — 1922	760	4.20 K	15.17 ..	0.1274
	370	3.55 ₀	14.85 ..	0.1248
	350	3.50 ₈	10.37 ..	0.0871
	345	3.49 ₆	7.81 ..	0.0656
	340	3.48 ₄	4.41 ..	0.0371
	330	3.46 ₂	0.80 ..	0.0067
	310	3.41 ₆	0.00 ..	0.0000
29 — VI — 1922		roomtemp.	121 Ω	1.
16 — VI — 1922	420	3.65 ₂ K	15.81 ..	0.1307
	400	3.61 ₂	15.81 ..	0.1307
	380	3.57 ₀	15.64 ..	0.1293
	370	3.55 ₀	15.32 ..	0.1266
	360	3.52 ₈	13.46 ..	0.1112
	350	3.50 ₆	7.59 ..	0.0627
	340	3.48 ₄	1.02 ..	0.0084
	330	3.46 ₂	0.003 ..	0.0003
	320	3.44 ₀	0.000 ..	0.0000

But they were always lower than the "vanishing point" of ordinary tinwires, namely by about 0.1° to 0.2° K.

In the tables II, III and IV the results of these measurements are given.

To make it possible to unite them all in one figure (Fig. 1) the resistances of the films are all divided by the value of the resistance measured at room-temperatures, on the day before that on which the measurements in liquid helium were carried out. As a matter of course, the absolute accuracy of these quotients, in the figure and the tables called $\frac{W}{W_k}$, is small. The temperatures at which the resistance has half disappeared ("vanishing-point") are given in table V for these three films as well as for the resistance coil Sn-1922-M, made of extruded tinwire ¹⁾. The density of the current through the film is also given. This is calculated as if the current

¹⁾ Diss. W. TUYN, Leiden 1924 p. 8.

TABLE V.

Date	Name of the resistance	$T_{1/2}$	Density of the current
	Sn-1992-M	$3^{\circ}.74_6$ K.	
19 — V — 1922	T_{pI}	3.63_6	2.5 amp./ mm^2
9 — VI — 1922	T_{pII}	3.48_4	0.17 ..
16 — VI — 1922	$T_{pII'}$	3.49_6	0.25 ..
30 — VI — 1922	$T_{pII'}$	3.50_6	0.25 ..

distributes itself uniformly through the whole section of the film. The influence of the density is clear from the inclinations of the lines in fig. 1. The question arises whether the low "vanishing-point" is due to thinness of the film or to its granular structure. The fact that the vanishing-point of the film $T_{pII'}$ appeared to change with time supports the last suggestion. This change might be connected with the change in structure ("aging") which is clear from the increase of the resistance.

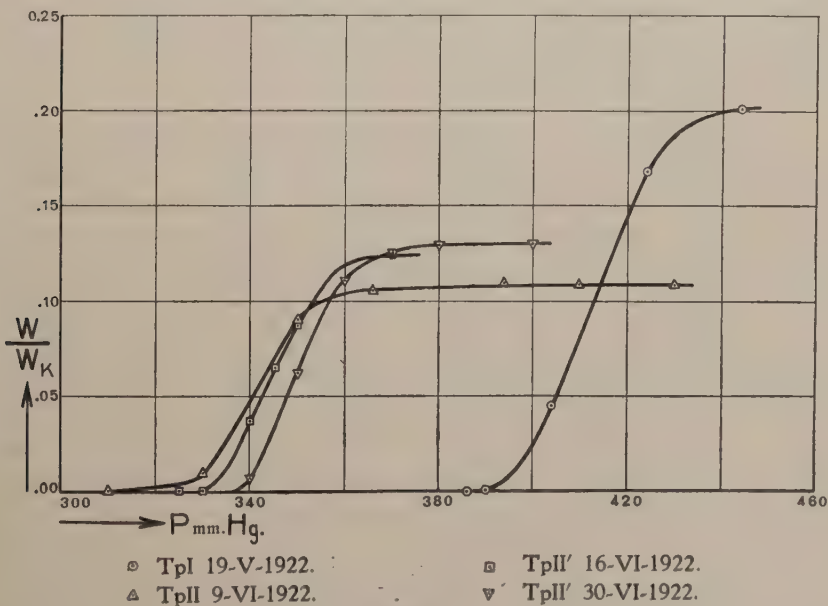


Fig. 1.

§ 3. With the tinfilm some measurements were further carried out concerning the disturbance of the supraconductivity by a magnetic field.

Though these investigations are only to be considered as of an orienting nature, they were a sufficient basis for concluding that the threshold value of the field for these films cannot greatly differ from that for tin

in the form of extruded wires. For example, with the film T_pII' a magnetic field of 55 gauss, directed parallel to the direction of the current through the film was sufficient to bring back the resistance of the film to the half value, at the temperature of 3°.42 K. For a time the required fieldstrength at this temperature amounts to about 45 gauss.

Lastly for one of the films the threshold value of the current could be determined roughly at two temperatures. It was found to be 0.16 Amp. at 3°.44 K. and 0.80 Amp. at 2°.78 K. The thickness of this film was about 0.5 μ , the breadth 15 mm., thus the section about 0.01 mm². So it was of the same order of magnitude as that of the tinwires, mentioned in Leyden Comm. No. 133d ¹⁾. For these tinwires threshold values of the current are given which are also of the same order of magnitude as those found with this tinfilm.

This result thus is not in agreement with the suggestion of SILSBEE ²⁾, namely that thin metallic films should have an abnormally high threshold value of the current, in consequence of the small value of the magnetic field of a current sent through a thin film.

Rather than to reject the hypothesis of SILSBEE about the connection of the threshold values of the current and the magnetic field, which is strongly supported by the measurements of W. TUYN ³⁾, we are apt to consider as false the supposition about the uniform distribution of the current through the film. Besides it is difficult to form an idea about the roll, which the granular structure of the film plays in this connection.

§ 4. Besides tin other supraconductors have not been investigated. Some measurements are made with bismuth films. Namely it was known ⁴⁾ that bismuth did not become supraconductive at 4°.25 K., but it had not yet been investigated at lower temperatures. As a bismuth film was easily to obtain while the drawing and winding of bismuth wires gives considerable difficulty because of the fragility of the wire, the resistance of some bismuth films was determined at helium temperatures. The material was electrolytic bismuth.

In the whole region of the temperatures obtainable with liquid helium the resistance of these films remained practically constant. Even at 1°.24 K. supraconductivity did not appear.

¹⁾ These Proceedings 16, 1914, p. 673.

²⁾ F. B. SILSBEE. Journal Wash. Ac. of Sc. Vol. VI. (1916) p. 597.

³⁾ Diss. Leiden 1924. Cap. II.

⁴⁾ Leyden Comm. N°. 142b (1914). These Proceedings 17, 1914. p. 520.

Physiology. — „Ueber die nervöse Natur und das Vorkommen der sogenannten interstitiellen Zellen (CAJAL, DOGIEL) in der glatten Muskulatur.“ (Aus dem Laboratorium für Embryologie und Histologie in Utrecht, Dir. Prof. J. BOEKE.) Von B. I. LAWRENTJEW, Kasan. (Communicated by Prof. J. BOEKE.)

(Communicated at the meeting of November 28, 1925).

In der vorliegenden Mitteilung will ich über einige interessante Tatsachen berichten, die ich bei der Untersuchung des feineren Baues des peripheren autonomen Nervensystems zu beobachten Gelegenheit hatte, und die meines Erachtens eine selbständige Bedeutung haben.

Es handelt sich um die sogen. interstitiellen Zellen von CAJAL und DOGIEL, deren Natur bis jetzt bei weitem nicht aufgeklärt war. Bekanntlich lassen sich in den Wandungen des Verdauungskanals vieler Wirbelthiere ausser den typischen Nervenzellen des AUERBACHSchen und MEISSNERSchen Plexus mit Hilfe von Methylenblau oder Silberimpregnation kleine spindelförmige oder dreieckige Zellen mit dünnen langen varikösen Fortsätzen nachweisen, die in grosser Anzahl zwischen den Muskelschichten, in der Periferie der Nervenganglien und Blutgefässe, in der Mucosa und Submucosa eingelagert sind. Diese von CAJAL im Jahre 1889 entdeckten Zellen waren seitdem stets der Gegenstand aufmerksamer Beobachtungen vieler Forscher. In der Literatur finden wir zwei sich widersprechende Ansichten über die Natur dieser Elemente.

RAMON Y CAJAL weist auf Grund seiner zahlreichen Beobachtungen und der Untersuchungen von LA WILLA auf das Vorhandensein eines neurofibrillären Apparates in den interstitiellen Zellen hin, der sich in einigen Fällen besser mit Silber imprägnieren lässt als in den typischen Zellen des AUERBACHSchen und MEISSNERSchen Plexus. Das Vorhandensein von Neurofibrillen, die variköse Form der Fortsätze und die mit den Nervenzellen übereinstimmende Färbbarkeit mit Methylenblau sind nach der Meinung von CAJAL und LA WILLA unzweifelhafte Beweise der nervösen Natur der interstitiellen Zellen. CAJAL und LA WILLA beschreiben diese Zellen als eine Reihe nicht mit einander verbundener Neurone — „neurones sympathiques interstitielles“. Die Fortsätze dieser Zellen scheinen nach CAJAL mit den glatten Muskelfasern und den LIEBERKÜHNschen Drüsen in Verbindung zu stehen.

Es muss aber bemerkt werden, dass es weder CAJAL, noch LA WILLA gelungen ist, einen Zusammenhang der interstitiellen Zellen mit den übrigen Nervelementen des sympathischen Nervensystems nachzuweisen. Ueber die Endigungen der Fortsätze der interstitiellen Zellen drückt sich CAJAL ebenfalls sehr vorsichtig aus (CAJAL 1911, S. 934).

Hierbei sei erwähnt, dass PAUL SCHULTZ in 1895 sehr genau und ausführlich interstitielle Zellen im Darm, in der Harnblase und in den Harnleitern des Frosches und vieler Säugethiere beschrieben hat. Er beobachtete die Endigungen der Fortsätze der interstitiellen Zellen in der glatten Muskulatur und spricht sich entschieden für die nervöse Natur dieser Zellen aus.

Nach ERIK MÜLLER sind die interstitiellen Zellen echte Nervelemente deren Ursprung er auf Grund seiner zahlreichen embryologischen Untersuchungen sogar feststellt. Die interstitiellen Zellen entstehen aus der Keimanlage des sympathischen Nervensystems, während die grosse Mehrzahl von Nervenzellen des AUERBACHSchen und MEISSNERSchen Geflechts im Magen und Dünndarm aus der Keimanlage des Nervus Vagus sich entwickeln. Im Magen der Embryone von *Squalis acantias*, sowie im Magen von Hühnerembryonen beobachtete MÜLLER alle Entwicklungsstadien dieser Zellen. Die langen Fortsätze dienen zur Anastomose der Zellen unter einander. Innerhalb der Zellen und ihrer Fortsätze verlaufen Neurofibrillen verzweigen sich in der glatten Muskulatur und in der Schleimhaut. Es gelang MÜLLER nicht an den interstitiellen Zellen einen pericellulären Apparat nachzuweisen. Nach MÜLLER sind die interstitiellen Zellen ein syncytiales System, das für den Darmtraktus spezifisch ist. Indem er diesem System eine spezielle physiologische Funktion zuschreibt, findet er in demselben viel Gemeinsames mit peripheren Nervennetzen, die BETHE 1895 in der Froschzunge beschrieb.

DOGIEL wies mit Hilfe der Methylenblaufärbung die interstitiellen Zellen im Darm, im Unterhautbindegewebe, in der Gallenblase, im Zentrum tendineum des Zwerchfells von Säugetieren nach und lieferte eine Reihe sehr schöner direkt klassischer Abbildungen. Nichtsdestoweniger konnte DOGIEL weder Neurofibrillen in diesen Zellen nachweisen noch einen Zusammenhang derselben mit den Nervenfasern feststellen. Auf Grund seiner Untersuchungen spricht er sich entschieden für die bindegewebige Natur der interstitiellen Zellen aus, ebenso wie HUBER.

MARTIN HEIDENHAIN schliesst sich nach Zusammenfassung der Ergebnisse dieses Streites vollkommen der Ansicht DOGIELS an, indem er erklärt, dass gar kein Grund vorliege, den interstitiellen Zellen den Charakter von Nervelementen zuzuschreiben.

Bei Untersuchungen über den Bau der Nervegeflechte in der glatten Muskulatur, welche ich in Anschluss an den Untersuchungen von VAN ESVELD über die Elemente des sympathischen Plexus von AUERBACH und MEISSNER anstellte, erzielte ich ebenso wie VAN ESVELD mittels Methylenblaufärbung wiederholt sehr schöne Bilder der interstitiellen Zellen und war mit DOGIEL geneigt, dieselben für Bindegewebszellen zu halten, zumal nach Kenntnisnahme der Arbeit TELLO's, der argentophile Fasern in Bindegewebszellen nachwies. Erst nach Anwendung der BIELSCHOWSKY Methode in der Modifikation von GROSS war ich genötigt, meine Ansicht über die interstitiellen Zellen radikal zu ändern.

Als Untersuchungsobjekt dienten mir die Speiseröhre, der Magen, Darm und die Harnblase von Ratten, Mäusen, Kaninchen und Katzen.

Bij gelungener Impregnation nach der BIELSCHOWSKY-GROSS Methode erhält man sehr schöne Bilder der Nervenverzweigung in der glatten Muskulatur, in der Mukosa und Submukosa der erwähnten Organe. Fertigt man einen Flächenschnitt im Gebiet der Submukosa des Magens oder Darms an, so sieht man bei mässiger, sogenannter elektiver Impregnation das vielfach beschriebene Geflecht schichtweise über einander gelagerter markloser Nervenfasern, die bis in die Mukosa reichen. In jedem Nervenstämmchen kann man scharf imprägnierte Nervenfasern von verschiedener Dicke sehen, die meist ausserordentlich dünn sind. Sehr charakteristisch ist die Lage der Kerne in diesen Nervenstämmchen, die nicht ausserhalb, sondern innerhalb dieser Nervenstämmchen liegen, so dass die Nervenfasern (Neurofibrillenbündel) die Kerne von allen Seiten umgeben. Dasselbe Bild kann man mittels jeder beliebigen Silberimpregnationsmethode erhalten. (CAJAL, GOLGI, RANSON usw.) Der Vorzug der Methode BIELSCHOWSKY's besteht hauptsächlich darin, dass man die Imprägnierung nach Belieben verstärken kann, und dann treten sehr interessante Einzelheiten hervor. Betrachtet man z.B. den Kreuzungspunkt der geschilderten Nervenstämmchen, so kann man, wie aus der Abbildung (I) zu ersehen ist, beobachten, dass die Nervenfasern (Neuro-

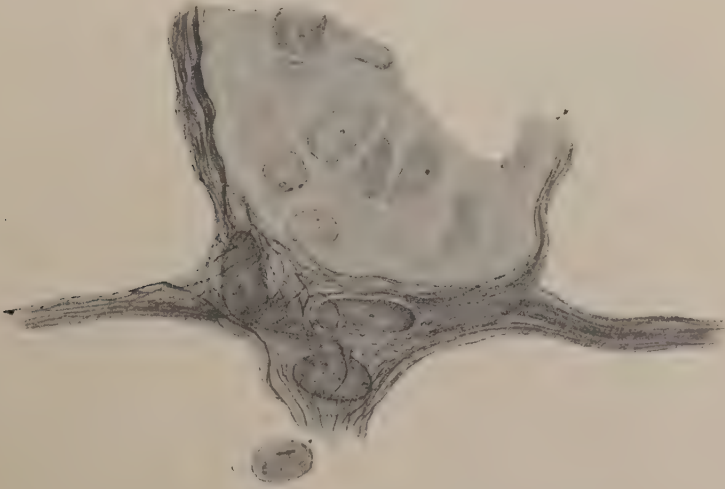


Abb. 1.

Kaninchen, Magen, Submukosa. Kreuzungspunkt des Nervenengeflechtes. BIELSCHOWSKY.

fibrillenbündel) innerhalb des Protoplasmageflechtes liegen, von dem nach allen Richtungen Protoplasmastränge verlaufen. Das Protoplasma dieser Stränge ist mit Silber imprägniert, was doch nicht hindert, den Verlauf der Zarten Nervenfasern zu verfolgen. Es ist sehr merkwürdig, dass an einem solchen Kreuzpunkte die Nervenfasern (Neurofibrillen-

bündel) nicht direkt aus einem Stämmchen ins andere übergehen, sondern häufig einen verwickelten Weg einschlagen: sie machen wiederholte Biegungen, kehren zu ihrem Ausgangspunkt um und bilden um die Kerne dieser Protoplasmaanhäufung eine Art Flechtwerk. Ob hierbei ein wirkliches Netz entsteht, oder ob sich die Nervenfäden nur kreuzen, ist ausserordentlich schwer zu entscheiden. Es ist nur vollkommen klar, dass die Nervenfäden (Neurofibrillenstränge) innerhalb des Protoplasmas liegen. Es kann hier von einer Anlagerung der Neurofibrillen oder Axone an die SCHWANNschen Zellen keine Rede sein, denn wie sollten sich bei einer blossen Anlagerung die Geflechte um die Kerne herum bilden? Die Imprägnierung kann noch verstärkt werden; dann wird das Protoplasma der Stränge und der Kreuzungspunkte noch dunkler, die zarte wabige Struktur des Protoplasmas wird gut sichtbar und die Nervenfäden werden kaum erkennbar.

Man kann schliesslich durch Wechseln der Intensität der Imprägnierung alle möglichen Uebergänge erzielen — von einer scharfen Imprägnierung der Nervenfäden allein bis zur Imprägnierung des Protoplasmas und an demselben Geflecht bald das imprägnierte Protoplasma, bald die in demselben liegenden Nervenfäden beobachten.

Somit sehen wir in der Submukosa und Mukosa des Magens und Darmes ein Protoplasmasyncytium, in welchem Nervenfäden eingelagert sind. Ich gebrauche absichtlich die Bezeichnung *Nervenfäden*, um den Vorwurf zu entgehen, den MARTIN HEIDENHAIN der Auffassung HELD's macht. Die Frage ob wir hier einen Komplex von Neurofibrillen oder ein sogen. Kabelsystem vor uns haben, bin ich zur Zeit nicht imstande zu beantworten. Es ist wohl möglich dass hier, wie HERINGA bei der Untersuchung der Nervenendigungen am Vogelschnabel gezeigt hat, eine Unterscheidung der einzelnen Axone nicht gelingt, so dass man in diesem Syncytium nur von miteinander zusammenliegenden, wahrscheinlich aus verschiedenen Axonen herkömmlichen Neurofibrillenbündeln sprechen konnte ¹⁾. Fast dasselbe Verhalten finden wir in der Muscularis propria der Speiseröhre, des Magens, des Darmes und Muscularis der Harnblase. Längs den dünnen bindegewebigen Scheidewänden verlaufen Nervenstämmchen, von denen Zweige zur glatten Muskulatur abgehen. Auch hier sind diese Stämmchen nichts anderes als Protoplasmastränge, zuweilen von flacher Form, in denen Kerne eingelagert sind, die immer innerhalb des Stranges liegen.

Die für die Mukosa und Submukosa typischen Kreuzungspunkten finden wir hier nicht vor. Von diesen Nervenstämmchen gehen meist unter einem rechten Winkel feinste Bündel von Nervenfäden ab, die zu den glatten Muskelzellen ziehen. Bei genügender Imprägnierung kann man auch hier ganz klar sehen, dass die Fäden innerhalb eines sehr feinen Stranges liegen, der die unmittelbare Fortsetzung des Grundstämmchens

¹⁾ HERINGA schlägt für einen solchen Fall den Namen Neuroplasmabahn vor.

bildet. Die Protoplasmaschicht ist hier sehr dünn, im weiteren Verlauf wird sie aber wieder dicker und hier sehen wir einen ovalen oder zuweilen auch fast runden Kern. Diese Kerne lassen sich gewöhnlich stärker imprägnieren als die Kerne von den glatten Muskelfasern und nehmen meist eine intensiv schwarze Farbe an. Die Nervenfasern gehen um den Kern von der einen oder von beiden Seiten herum und verzweigen sich dann nach allen Seiten unter einem weit stumpfen Winkel, indem sie aber stets innerhalb des feinen Protoplasmafortsatzes bleiben, und sobald sie eine bestimmte glatte Muskelfaser erreichen, bilden sie in derselben eine motorische Endigung. Ist das Protoplasma um einen solchen Kern genügend intensiv imprägniert, so hat das ganze Gebilde das Aussehen einer spindelförmigen oder dreieckigen Zelle mit feinen langen nach verschiedenen Richtungen verlaufenden Fortsätzen, d.h. wir haben eine typische interstitielle Zelle CAJAL's und DOGIEL's vor uns!

Ebenso wie in der Submukosa gelang es mir auch hier alle möglichen Abstufungen der Imprägnierung zu erzielen. Man kann nur die Neurofibrillen (meines Erachtens sind wir jetzt durchaus berechtigt, von Neurofibrillen zu sprechen) imprägnieren, dann sieht man nur, dass sie dicht an den Kern herankommen und ihn umgeben. Man kann auch gleichzeitig das Protoplasma und die Neurofibrillen imprägnieren, wie auf der Abbildung (2) zu sehen ist. Ebenso wie an den Kreuzungspunkten der

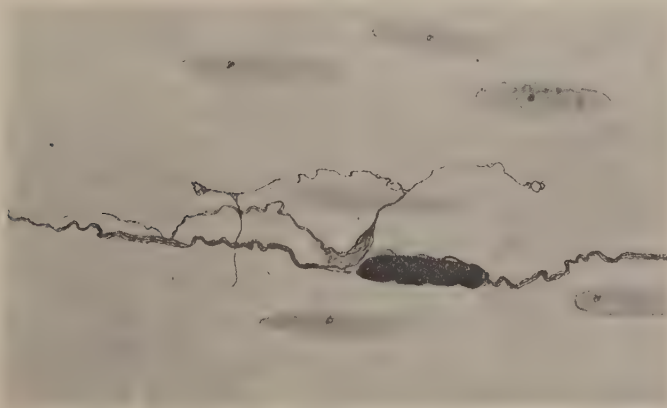


Abb. 2.

Harnblase, Ratte, *T. muscularis*.

Interstitielle Zelle. Man kann intraprotoplasmatische Neurofibrillen bis zur Endigung verfolgen. BIELSCHOWSKY-Meth.

Submukosa gehen die Neurofibrillen häufig nicht direkt in die Fortsätze über, sondern bilden eine Art Geflecht. Bei starker Imprägnierung endlich, bei der das wabige Protoplasma sich dunkel färbt, sehen wir das typische Bild der interstitiellen Zellen, wie sie mittels der Methylenblaufärbung dargestellt werden und u.a. von VAN ESVELD beschrieben werden.

DOGIEL hat vollkommen recht — die interstitiellen Zellen anastomosieren mit einander mittels ihrer Protoplasmafortsätze, ebenso recht hat aber auch RAMON Y CAJAL wenn er sagt, dass im Protoplasma der Zellen und in ihren Fortsätzen Neurofibrillen enthalten sind, deren Verlauf sich leicht bis zu den motorischen Endigungen in der glatten Muskelfasern verfolgen lässt. Die intraplasmatische Lage der Neurofibrillen lässt sich

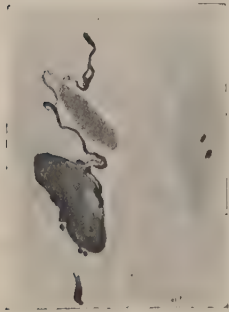


Abb. 3.

Ratte, Harnblase.

In dem Protoplasma der grossen interstitiellen Zelle quergeschnittene Neurofibrillenbündel als Pünktchen sichtbar.

BIELSCHOWSKY-Meth.

vollkommen klar auf Querschnitten beweisen wie aus Abbildung (3) zu ersehen ist, wo man sieht, dass die Neurofibrillenbündel mitunter so nahe dem Kerne anliegen, dass in demselben zuweilen eine Einbüchtung zu bemerken ist.

In dieser kurzen Mitteilung übergehe ich eine ganze Reihe Einzelheiten ebenso wie einen genauen Vergleich der erhaltenen Bilder mit den Angaben in der Literatur, indem ich hoffe, zu dieser Frage in einer anderen Arbeit über die Innervation der glatten Muskulatur, die demnächst veröffentlicht werden soll, zurückzukehren.

Hier wollte ich nur in Anschluss an die ebenfalls in diesen Proceedings beschriebenen Mitteilungen VAN ESVELD's über die Elemente des sympathischen Plexus von AUERBACH und MEISSNER meine Beobachtungen über die interstitiellen Zellen kurz referieren.

Ich beschränke mich nur auf folgende kurze Zusammenfassung der Ergebnisse.

In der glatten Muskulatur, in der Mukosa und Submukosa des Verdauungskanals der Säugetiere finden wir grosse protoplasmatische Syncytien, in denen ein System von Neurofibrillen enthalten ist.

Die Protoplasmaanhäufungen mit ihren Kernen und Strängen bilden ein abgeschlossenes und ununterbrochen verlaufendes Gebilde — Lemmoblasten nach der Terminologie HELD's — bis zu den terminalen Nervenendigungen. Die interstitiellen Zellen sind das letzte Glied dieser Leitungsbahn; ihre Fortsätze begleiten die Neurofibrillen bis zu den Endverzweigungen und Endigungen in den glatten Muskelfasern. Es ist durchaus verständlich, dass diese Endglieder, diese eigentümlichen Lemmoblasten überall dort gefunden werden, wo Endigungen des autonomen Nervensystems vorhanden sind: in der glatten Muskulatur, in den Drüsen und Gefässen, wie DOGIEL tatsächlich nachgewiesen hat. Ich bin vollkommen überzeugt, dass vollkommen analoge Bilder des Verlaufs der Neurofibrillen innerhalb der interstitiellen Zellen sich in jedem beliebigen Organ, im unmittelbaren Zusammenhang mit den Endigungen des autonomen Nervensystems werden darstellen lassen. Wir sehen auch, dass die in der letzten Zeit durch die Arbeiten BOEKE's und HERINGA's

glänzend weiter entwickelte Auffassung HELD's, nach der die Neurofibrillen der Spinalnerven bis zu ihren terminalen Verzweigungen in einer Kette von Lemmoblasten verlaufen, eine schöne Bestätigung durch die Art der Nervenendigungen des autonomen Nervensystems findet.

Wie sind die oben geschilderten Kreuzungspunkte und die interstitiellen Zellen zu deuten, in denen, wie wir sehen, die Neurofibrillen einen recht komplizierten Weg durchmachen, indem sie eine Art von Geflecht bilden? Wird in diesen Punkten die Nervenversorgung verändert, sind diese Bahnen nur einfache Reizleitungen? Ich bin vollkommen überzeugt, dass diese hochinteressante und wichtige Frage durch das Experiment zu lösen ist, an das ich jetzt herantreten werde.

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Mathematics. — „Sur les points accessibles des ensembles fermés” ¹⁾.

By PAUL URYSOHN †. (Communicated by Prof. L. E. J. BROUWER).

(Communicated at the meeting of November 28, 1925).

1. Désignons par E_n l'espace euclidien à n dimensions et soit F un ensemble fermé quelconque situé dans E_n . Nous dirons qu'un point a de F est accessible s'il existe dans E_n un arc simple \overline{ba} , aboutissant au point a et dont tous les autres points sont étrangers à F . Nous dirons que a est accessible au sens étroit s'il est possible de supposer, dans la définition précédente, que \overline{ba} soit un segment rectiligne.

Le but de cette note est de démontrer la proposition suivante.

THÉORÈME. — *Quel que soit l'ensemble fermé F situé dans l'espace E_n , l'ensemble L resp. L_d des points de F qui sont accessibles, resp. accessibles au sens étroit, cet ensemble est toujours un ensemble (A) de Souslin ²⁾.*

Nous montrerons en outre par des exemples que (au moins au cas $n \geq 3$) ce théorème ne peut pas être précisé davantage.

2. Commençons par prouver que l'ensemble L est toujours un ensemble (A).

Désignons comme d'habitude par $S(\xi, \varepsilon)$ l'intérieur de la sphère (n -dimensionnelle) de centre ξ et du rayon ε . Nous dirons que la sphère $S(\xi, \varepsilon)$ est canonique si elle vérifie les conditions suivantes:

1^o toutes les coordonnées du point ξ et le nombre ε sont rationnels,

2^o ξ appartient à $E_n - F$.

3^o $S(\xi, \varepsilon)$ a des points communs avec F .

L'ensemble de toutes les sphères canoniques étant évidemment dénombrable, rangeons-les d'une façon quelconque en une suite bien déterminée:

$$(1) \quad S_1, S_2, \dots, S_{i_1}, \dots$$

(i_1 étant un entier positif quelconque).

Supposons construites toutes les sphères $S_{i_1 i_2 \dots i_k}$, où i_1, i_2, \dots, i_k prennent indépendamment l'un de l'autre toutes les valeurs entières et positives.

Une $S_{i_1 i_2 \dots i_k}$ étant donnée, soient

$$(2) \quad S_{i_1 i_2 \dots i_k 1}, \quad S_{i_1 i_2 \dots i_k 2}, \text{ etc.} \quad S_{i_1 i_2 \dots i_k i_{k+1}}, \text{ etc. in inf.}$$

¹⁾ Les résultats formant le travail ci-dessus furent établis par PAUL URYSOHN dans les derniers mois de l'année 1923. Ils ont été communiqués à cette même époque à la Société Mathématique de Moscou. La présente rédaction a été faite par PAUL ALEXANDROFF (Moscou); les figures sont dues à D. VAN DANTZIG (Amsterdam).

²⁾ Voir pour la définition des ensembles (A) la note de SOUSLIN (Comptes Rendus t. 164, séance du 8 janvier 1917).

toutes les sphères canoniques contenues dans $S_{i_1 i_2 \dots i_k}$ et vérifiant en outre les deux conditions que voici:

4^o le rayon $r_{i_1 i_2 \dots i_{k+1}}$ de $S_{i_1 i_2 \dots i_{k+1}}$ est inférieur à la moitié du rayon $r_{i_1 i_2 \dots i_k}$ de $S_{i_1 i_2 \dots i_k}$

5^o les centres $c_{i_1 i_2 \dots i_k}$ et $c_{i_1 i_2 \dots i_{k+1}}$ de $S_{i_1 i_2 \dots i_k}$ et $S_{i_1 i_2 \dots i_{k+1}}$ peuvent être joints par une ligne polygonale $\Delta_{i_1 i_2 \dots i_{k+1}}$ située dans

$$(E_n - F) \cdot S_{i_1 i_2 \dots i_k}.$$

Les sphères $S_{i_1 i_2 \dots i_k}$ se trouvent ainsi définies par induction quels que soient les nombres naturels k, i_1, i_2, \dots, i_k . Désignons par M l'ensemble (A) défini par le système déterminant $\{S_{i_1 i_2 \dots i_k}\}$.

Démontrons que l'ensemble M coïncide avec L .

Soit a un point quelconque de M . Il existe, d'après la définition de M au moins une suite d'entiers positifs

$$i_1, i_2, \dots, i_k, \dots$$

telle que

$$(3) \quad a = \prod_{k=1}^{\infty} S_{i_1 i_2 \dots i_k}.$$

L'ensemble

$$(4) \quad a + \sum_{k=3}^{\infty} \Delta_{i_1 i_2 \dots i_k}$$

est évidemment fermé; il a avec $\Delta_{i_1 i_2}$ au moins le point $c_{i_1 i_2}$ en commun. Soit d_1 le premier point de l'ensemble (4) qu'on rencontre en parcourant $\Delta_{i_1 i_2}$ dans le sens de c_{i_1} à $c_{i_1 i_2}$. Désignons par Π_1 le segment $\overline{c_{i_1} d_1}$ du polygone $\Delta_{i_1 i_2}$. Le point d_1 est nécessairement différent de a (puisque $a \in F$ en même temps que $d_1 \in \Delta_{i_1 i_2} \subset E_n - F$); il en résulte que d_1 appartient au moins à un $\Delta_{i_1 i_2 \dots i_k}$, $k \geq 3$.

Supposons que nous ayons construit une ligne polygonale simple $\Pi_s = \overline{c_{i_1} d_s} \subset E_n - F$ de façon que $d_s \neq a$ est le seul point commun à Π_s et à l'ensemble fermé

$$(4_s) \quad a + \sum_{k=h(s)}^{\infty} \Delta_{i_1 i_2 \dots i_k}$$

où $h(s)$ est un certain entier non inférieur à s (pour $s=1$ on a $h(1)=3$). Comme d_s est différent de a , il existe un premier indice $k_0 \geq h(s)$ tel que

$$d_s \in \Delta_{i_1 i_2 \dots i_{k_0}} - c_{i_1 i_2 \dots i_{k_0}}.$$

Considérons l'ensemble fermé

$$(4'_s) \quad a + \sum_{k=k_0+1}^{\infty} \Delta_{i_1 i_2 \dots i_k}$$

ayant avec $\Delta_{i_1 i_2 \dots i_{k_0}}$ au moins le point $c_{i_1 i_2 \dots i_{k_0}}$ en commun.

Soit d_{s+1} le premier point de l'ensemble $(4'_s)$ qu'on rencontre sur $\Delta_{i_1 i_2 \dots i_{k_0}}$ en allant de d_s à $c_{i_1 i_2 \dots i_{k_0}}$. Comme $\Delta_{i_1 i_2 \dots i_{k_0}}$ est étranger à F ,

d_{s+1} appartient à un au moins $A_{i_1 i_2 \dots i_k}$, $k \geq k_0 + 1$. Soit $h(s+1)$ le plus petit parmi les nombres $k \geq k_0 + 1$ tels que $d_{s+1} \in A_{i_1 i_2 \dots i_k}$. Posons

$$(5) \quad \Pi_{s+1} = \Pi_s + \overline{d_s d_{s+1}}$$

où $\overline{d_s d_{s+1}}$ est le segment de $A_{i_1 i_2 \dots i_{k_0}}$ aux extrémités d_s et d_{s+1} .

Il est évident que d_{s+1} est le seul point commun à Π_{s+1} et à l'ensemble

$$(4_{s+1}) \quad a + \sum_{k=h(s+1)}^{\infty} A_{i_1 i_2 \dots i_k}$$

on a en outre

$$h(s+1) \leq k_0 + 1 \leq h(s) + 1 \leq s + 1.$$

En procédant ainsi par induction on obtient une suite infinie de lignes polygonales simples:

$$\Pi_1 \subset \Pi_2 \subset \dots \subset \Pi_s \subset \dots, \quad \Pi_s = c_{i_1} \overline{d_s} \subset E_n - F$$

Comme le diamètre de $\sum_{j=1}^{\infty} \overline{d_j d_{j+1}}$ et la distance $\varrho(d_s, a)$ deviennent infiniment petits avec $\frac{1}{s}$, l'ensemble fermé $a + \sum_{s=1}^{\infty} \Pi_s$ est immédiatement homéomorphe à un segment rectiligne; il n'a d'ailleurs avec F que le seul point a en commun. Ce dernier point est donc accessible et l'inclusion $M \subset L$ est ainsi démontrée.

3. Pour démontrer l'inclusion inverse $L \subset M$, considérons un point accessible quelconque de l'ensemble F . Soit a ce point et $\overline{ab_0}$ un arc simple agrégé à $(E_n - F) + a$.

On peut toujours supposer que $\overline{ab_0}$ soit une „ligne polygonale généralisée” c. à d. que

$$(7) \quad \overline{ab_0} = a + \sum_{i=0}^{\infty} \overline{b_i b_{i+1}}$$

où $\overline{b_i b_{i+1}}$ sont des segments rectilignes convergents vers le point a (donc de longueur tendant vers 0 avec $\frac{1}{i}$) et tels que $\overline{b_i b_{i+1}}$ et $\overline{b_k b_{k+1}}$, $i < k$, n'ont aucun point commun, si $k > i + 1$, et ont le seul point b_{i+1} en commun, si $k = i + 1$.¹⁾

En déplaçant légèrement les points b_i , il est facile d'obtenir un arc

¹⁾ Soient en effet, $\overline{ab_0}$ un arc simple quelconque n'ayant avec F que le point a en commun, et

$$b_0, b_1, b_2, \dots, b_n, \dots$$

une suite infinie des points de $\overline{ab_0}$ tendant vers a . On peut évidemment joindre b_i et b_{i+1} par une ligne polygonale $\Lambda_{i+1} \subset E_n - F_1$ très voisine de l'arc correspondant $\overline{b_i b_{i+1}}$ de $\overline{ab_0}$.

Un raisonnement absolument analogue à celui du numéro précédent permet alors d'extraire du continu $a + \sum_{i=1}^{\infty} \Lambda_i$ un arc simple de la forme (7).

simple de la forme (7) dont les „sommets” b_i ($i = 0, 1, 2, \dots$) ont toutes leurs coordonnées rationnelles. Nous supposons donc que (7) vérifie cette condition supplémentaire. Désignons enfin par $\overline{ab_i}$ le segment $\overline{ab_i}$ de l'arc simple (7).

Considérons maintenant une sphère $S(b_0, \varepsilon_0)$ d'un rayon rationnel ε_0 surpassant le diamètre de $\overline{ab_0}$. La sphère $S(b_0, \varepsilon_0)$ étant évidemment canonique, elle est une certaine S_{i_1} de la suite (1).

Supposons choisie une $S_{i_1 i_2 \dots i_k}$ telle que les deux conditions suivantes soient vérifiées :

1). $c_{i_1 i_2 \dots i_k}$ est un „sommets” b_{m_k} déterminé de la „ligne polygonale généralisée” (7).

2). $S_{i_1 i_2 \dots i_k} \supset \overline{ab_{m_k}}$.

Soit alors m_{k+1} le premier entier vérifiant la condition :¹⁾

$$(8) \quad \delta(\overline{ab_{m_{k+1}}}) < \frac{1}{2} \varrho(\overline{ab_{m_k}}, E_n - S_{i_1 i_2 \dots i_k})$$

Désignons par ε_{k+1} un nombre rationnel quelconque supérieur à la première et inférieur à la seconde partie de l'inégalité (8).

$S(b_{m_{k+1}}, \varepsilon_{k+1})$ est une sphère canonique contenue dans $S_{i_1 i_2 \dots i_k}$ puisque ε_{k+1} est inférieur à

$$\frac{1}{2} \varrho(\overline{ab_{m_k}}, E_n - S_{i_1 i_2 \dots i_k}) \leq \frac{1}{2} \varrho(b_{m_{k+1}}, E_n - S_{i_1 i_2 \dots i_k}).$$

On a de plus

$$\varepsilon_{k+1} < \frac{1}{2} \varrho(\overline{ab_{m_k}}, E_n - S_{i_1 i_2 \dots i_k}) \leq \frac{1}{2} \varrho(b_{m_k}, E_n - S_{i_1 i_2 \dots i_k}) = \frac{1}{2} r_{i_1 i_2 \dots i_k}$$

(où $r_{i_1 i_2 \dots i_k}$ est le rayon de $S_{i_1 i_2 \dots i_k}$).

Le segment $\overline{b_{m_k} b_{m_{k+1}}}$ de l'arc simple (7) est une ligne polygonale joignant le centre de $S_{i_1 i_2 \dots i_k}$ avec celui de $S(b_{m_{k+1}}, \varepsilon_{k+1})$. Cette ligne polygonale est étrangère à F ; d'après 2) elle est contenue dans $S_{i_1 i_2 \dots i_k}$. Les deux conditions 4° et 5° sont donc réalisées par la sphère $S(b_{m_{k+1}}, \varepsilon_{k+1})$, de sorte qu'elle est une $S_{i_1 i_2 \dots i_k i_{k+1}}$ déterminée. $S_{i_1 i_2 \dots i_k i_{k+1}}$ réalise enfin les deux conditions 1) et 2) ci-dessus : en effet, son centre est le point $b_{m_{k+1}}$; d'après la définition de ε_{k+1} , la sphère $S_{i_1 i_2 \dots i_k i_{k+1}}$ contient de plus l'arc simple $\overline{ab_{m_{k+1}}}$.

En procédant ainsi par induction on obtient une „chaîne régulière”

$$S_{i_1}, S_{i_1 i_2}, \dots, S_{i_1 i_2 \dots i_k}, \dots$$

de sphères canoniques auxquelles le point a est intérieur.

¹⁾ Où $\varrho(A, B)$ désigne, d'une façon générale, la distance entre les deux ensembles A et B , tandis que $\delta(C)$ est le diamètre de l'ensemble C .

On a donc

$$a \in \prod_{k=1}^{\infty} S_{i_1 i_2 \dots i_k}$$

ce qui exprime que le point a fait partie de l'ensemble M . L'inclusion $L \subset M$ et par suite l'identité $L = M$ se trouve ainsi démontrée. L'ensemble L est donc un ensemble (A).

4. Démontrons maintenant que l'ensemble L_d est lui aussi un ensemble (A). Comme L_d est l'ensemble de tous les points a de F accessibles par des segments rectilignes \overline{ab} ($\overline{ab} \cdot F = a$), on a évidemment l'identité

$$L_d = \sum_{\mu=1}^{\infty} L_d^{\mu},$$

où L_d^{μ} est l'ensemble de ceux-là parmi les points a de L_d , pour lesquels il existe un segment \overline{ab} de longueur $\geq \frac{1}{\mu}$.

Il suffit donc de prouver que tout ensemble L_d^{μ} est un ensemble (A).

Pour arriver à ce but, appelons „paire canonique” $\mathfrak{P} = (P, T)$ un couple de parallélépipèdes n -dimensionaux rectangulaires P et T jouissant des propriétés suivantes:

1°. Les sommets de P et de T ont toutes leurs coordonnées rationnelles.

2°. P et T sont adjacents, c. à d. ils ont une face $(n-1)$ -dimensionnelle et ils n'ont aucun autre point en commun. L'arête de T perpendiculaire à cette face commune est de longueur $\geq \frac{1}{\mu}$.

3°. P possède des points communs avec F tandis que T est situé dans $E_n - F$.

L'ensemble de toutes les paires canoniques étant évidemment dénombrable, nous le supposons rangé en une suite bien déterminée

(9) $\mathfrak{P}_1, \mathfrak{P}_2, \dots, \mathfrak{P}_{i_1}, \dots$ (i_1 est un nombre naturel quelconque).

Nous supposons d'ailleurs que \mathfrak{P}_{i_1} soit formé de P_{i_1} et T_{i_1} , tout en remarquant qu'un même parallélépipède P ou T peut figurer plusieurs fois respectivement dans la suite des P_{i_1} et dans la suite des T_{i_1} .

Supposons choisies, parmi les paires (9), toutes les paires $\mathfrak{P}_{i_1 i_2 \dots i_k} = (P_{i_1 i_2 \dots i_k}, T_{i_1 i_2 \dots i_k})$, où i_1, i_2, \dots, i_k prennent toutes les valeurs entières et positives. (Pour $k=1$ on a toutes les paires (9)).

Choisissons une $\mathfrak{P}_{i_1 i_2 \dots i_k}$ déterminée et considérons toutes les paires canoniques $\mathfrak{P} = (P, T)$ telles que les relations suivantes aient lieu simultanément:

$$\left. \begin{aligned} (10) \quad & \delta(P) < \frac{1}{2} \delta(P_{i_1 i_2 \dots i_k}) \\ (11) \quad & P \subset P_{i_1 i_2 \dots i_k} \\ (12) \quad & P + T \subset P_{i_1 i_2 \dots i_k} + T_{i_1 i_2 \dots i_k} \end{aligned} \right\}$$

Les $\mathfrak{P} = (P, T)$ ainsi définies seront désignés par

$$(13) \quad \mathfrak{P}_{i_1 i_2 \dots i_k 1}, \quad \mathfrak{P}_{i_1 i_2 \dots i_k 2}, \dots, \quad \mathfrak{P}_{i_1 i_2 \dots i_k i_{k+1}}, \dots$$

et on posera par définition

$$(14) \quad \mathfrak{P}_{i_1 i_2 \dots i_k i_{k+1}} = (P_{i_1 i_2 \dots i_k i_{k+1}}, T_{i_1 i_2 \dots i_k i_{k+1}}).$$

Désignons par N l'ensemble (A) défini par le système déterminant $\{P_{i_1 i_2 \dots i_k}\}$.

Je dis que N est identique à L_d^μ .

5. Soit d'abord a un point quelconque de N . Il existe alors une suite de nombres naturels $i_1, i_2, \dots, i_k, \dots$, telle que $a = \prod_{k=1}^{\infty} P_{i_1 i_2 \dots i_k}$. On voit de suite d'après la condition 2° du n° 4 et les relations (10) et (12), que la suite des parallélépipèdes

$S_{i_1}, S_{i_1 i_2}, \dots, S_{i_1 i_2 \dots i_k}, \dots$ (où l'on a posé $S_{i_1 i_2 \dots i_k} = P_{i_1 i_2 \dots i_k} + T_{i_1 i_2 \dots i_k}$)

est décroissante et admet comme partie commune un segment rectiligne A de longueur $\geq \frac{1}{\mu}$. D'après la condition 3° l'ensemble non vide $A.F$

est contenu, quel que soit k , dans $P_{i_1 i_2 \dots i_k}$; il se réduit donc au point a . Comme évidemment ce point n'est pas intérieur à A (en vertu de (10)), A est un segment \overline{ab} , dont tous les points à l'exception de a sont étrangers à F . Le point a est donc accessible par le segment A d'une longueur $\geq \frac{1}{\mu}$, c. à d. que $a \in L_d^\mu$ et par conséquent $N \subset L_d^\mu$.

6. Soit maintenant a un point quelconque de L_d^μ . Il existe un segment rectiligne $\overline{ab} \subset (E_n - F) + a$, d'une longueur $\geq \frac{1}{\mu}$. Si la longueur de \overline{ab} était précisément égale à $\frac{1}{\mu}$, on pourrait toujours prolonger un peu ce segment au-delà de b sans rencontrer aucun point de F . On peut donc supposer que la longueur de \overline{ab} est supérieure à $\frac{1}{\mu}$.

Cela posé, soit c un point de \overline{ab} assez voisin de a pour que \overline{cb} soit encore de longueur supérieure à $\frac{1}{\mu}$ (voir la fig. 1, faite pour le cas $n = 2$, d'ailleurs immédiatement applicable au cas général).

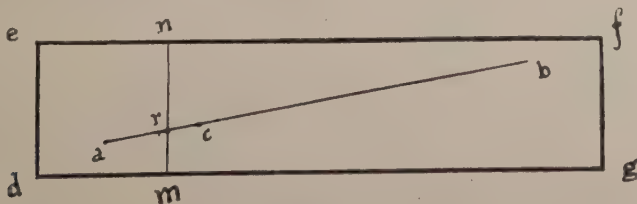


Fig. 1.

Il existe un parallélépipède „rationnel” (c. à d. dont les sommets ont toutes leurs coördonnées rationnelles), à n dimensions $defg$ (voir toujours la figure) contenant le segment \overline{ab} et situé tout entier dans un voisinage assez restreint de \overline{ab} pour qu'en menant par un certain point r de \overline{ac} , un hyperplan $(n-1)$ -dimensionnel mn perpendiculaire à l'arête dg on obtienne un parallélépipède $mngf$ dépourvu de points de F et dont l'arête mg a une longueur $\geq \frac{1}{\mu}$. On peut en outre supposer que $mngf$ et $denm$ soient des parallélépipèdes rationnels. On s'aperçoit aussitôt que les deux derniers parallélépipèdes forment une paire canonique, soit Ψ_{i_1} , avec $P_{i_1} = (denm)$ et $T_{i_1} = (mngf)$.

Supposons déjà construite une paire canonique

$$\Psi_{i_1 i_2 \dots i_k} = (P_{i_1 i_2 \dots i_k}, T_{i_1 i_2 \dots i_k})$$

telle que $S_{i_1 i_2 \dots i_k} = P_{i_1 i_2 \dots i_k} + T_{i_1 i_2 \dots i_k} \supset \overline{ab}$.

Il existe (voir la fig. 2).

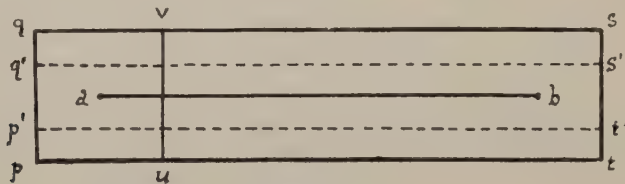


Fig. 2.

un parallélépipède $pqrst$ contenu dans $S_{i_1 i_2 \dots i_k}$ et assez voisin d' \overline{ab} pour qu'un hyperplan $(n-1)$ -dimensionnel uv , perpendiculaire à l'arête pt et assez rapproché de pq découpe de $pqrst$ un parallélépipède $pquv$ contenu dans $P_{i_1 i_2 \dots i_k}$ et de diamètre inférieur à $\frac{1}{2} \delta(P_{i_1 i_2 \dots i_k})$.

Nous supposons de plus que le parallélépipède $pqrst$ ait son arête \overline{pt} parallèle à la direction du segment \overline{ab} .

On peut évidemment supposer que le parallélépipède $vust$ soit étranger à l'ensemble F : dans le cas contraire on n'aurait en effet qu'à remplacer $pqrst$ par un parallélépipède $p'q's't'$ plus mince encore (c. à d. situé dans un voisinage plus restreint du segment \overline{ab}) et jouissant des mêmes propriétés par rapport à \overline{ab} .

Il ne nous reste que de déplacer légèrement nos parallélépipèdes afin de les transformer en parallélépipèdes possédant toutes les propriétés signalées et dont les sommets aient en outre toutes leurs coördonnées rationnelles. Une des propriétés en question sera en général mise en défaut par cette dernière opération: après son déplacement le parallélépipède $pqrst$ ne possèdera plus d'arête parallèle à \overline{ab} : il sera changé de direction, d'ailleurs aussi peu qu'on voudra.

Le couple des parallélépipèdes $pquv$ et $vust$ forme (comme on le

voit aussitôt) une paire canonique; cette paire canonique vérifie de plus les relations (10), (11), (12), elle est donc une $\mathfrak{P}_{i_1 i_2 \dots i_k i_{k+1}}$, avec $(pquv) = = P_{i_1 i_2 \dots i_k i_{k+1}}$ et $(vust) = T_{i_1 i_2 \dots i_k i_{k+1}}$. L'inclusion $S_{i_1 i_2 \dots i_k i_{k+1}} \supset \overline{ab}$ se trouve réalisée, ce qui permet de poursuivre indéfiniment le raisonnement d'induction ci-dessus.

On définit ainsi de proche en proche les nombres naturels

$$i_1, i_2, \dots, i_k, \dots$$

vérifiant la condition $a \in P_{i_1 i_2 \dots i_k}$; il en résulte que $a \in N$.

L'identité des deux ensembles N et L_d^μ , et par conséquent notre proposition entière, se trouvent ainsi démontrées.

7. Passons maintenant à la construction de quelques exemples se rattachant intimement au théorème que nous venons de démontrer.

Faisons d'abord la convention suivante. Nous désignerons dans ce n^0 par $I_{i_1 i_2 \dots i_k}$ (où k, i_1, i_2, \dots, i_k sont des nombres naturels quelconques) l'ensemble de tous les nombres irrationnels (positifs) dont le développement en fraction continue commence par

$$\frac{1}{i_1 + \frac{1}{i_2 + \frac{1}{i_3 + \dots + \frac{1}{i_k} + \dots}}}$$

Nous désignerons aussi par $a_{i_1 i_2 \dots i_k}$ et $b_{i_1 i_2 \dots i_k}$ respectivement la borne inférieure et supérieure de l'ensemble $I_{i_1 i_2 \dots i_k}$.

Supposons maintenant donné, dans l'espace tridimensionnel ordinaire, un système de coordonnées cylindriques z, r, φ .

Considérons sur l'axe Oz un ensemble (A) non-mesurable B quelconque agrégé à l'intervalle $0 < z < 1$ et défini par un système déterminant formé d'intervalles $\Delta_{i_1 i_2 \dots i_k} = (a_{i_1 i_2 \dots i_k}, b_{i_1 i_2 \dots i_k})$. Nous supposerons de plus (ce qui est possible pour tout ensemble (A)) qu'on ait toujours $\Delta_{i_1 i_2 \dots i_k} \supset \Delta_{i_1 i_2 \dots i_{k+1}}$ et que la longueur de l'intervalle $\Delta_{i_1 i_2 \dots i_k}$ tende vers 0 avec $\frac{1}{k}$.

Désignons par H le domaine formé de tous les points dont les coordonnées vérifient l'ensemble des inégalités suivantes:

$$(15) \quad \frac{1}{2} < r < 1; \quad 0 < \varphi < 1; \quad 0 < z < 1$$

De même, quels que soient les entiers positifs k, i_1, i_2, \dots, i_k , désignons par $H_{i_1 i_2 \dots i_k}$ l'ensemble de tous les points dont les coordonnées vérifient simultanément les relations

$$(16) \quad \frac{1}{k+2} < r \leq \frac{1}{k+1}, \quad a_{i_1 i_2 \dots i_k} < \varphi < b_{i_1 i_2 \dots i_k}, \quad a_{i_1 i_2 \dots i_k} < z < b_{i_1 i_2 \dots i_k}.$$

Posons

$$(17) \quad G = H + \sum_{k=1}^{\infty} \left(\sum_{i_1=1}^{\infty} \dots \sum_{i_k=1}^{\infty} H_{i_1 i_2 \dots i_k} \right),$$

$$(18) \quad F = E_3 - G.$$

On voit immédiatement que F est un ensemble fermé. Une analyse facile montre que les deux ensembles L et L_d coïncident dans notre cas et que l'ensemble de ceux-là parmi les points accessibles de l'ensemble F , qui sont situés sur l'axe Oz est précisément l'ensemble (A) que nous venons de placer sur cette axe. Par conséquent, la partie commune à l'ensemble L et à l'ensemble de tous les points de l'axe Oz est non-mesurable B ; il en résulte que l'ensemble $L = L_d$ est lui-même non-mesurable B .

Nous avons ainsi construit un exemple d'un ensemble fermé situé dans l'espace ordinaire et pour lequel les deux ensembles L et L_d sont non-mesurables B .

La question, si un pareil ensemble existe dans le plan, reste ouverte ¹⁾. Ainsi, du moins pour le cas $n \geq 3$, on ne peut pas remplacer dans l'énoncé de notre théorème l'affirmation que L ou L_d soient des ensembles (A) , par une affirmation plus précise.

8. Notre théorème ne pouvant pas être précisé, on pourrait peut-être songer à le généraliser, c. à d. à appliquer le même énoncé à une classe d'ensembles plus vaste que celle des ensembles fermés. Pour montrer que cela aussi est impossible, nous allons donner un exemple d'un ensemble G_δ pour lequel l'ensemble des points accessibles resp. accessibles au sens étroit n'est point un ensemble (A) .

Cette construction se fera d'ailleurs dans le plan euclidien E_2 .

Supposons donné dans le plan E_2 un système xOy de coordonnées rectangulaires. Désignons par Φ l'ensemble de tous les points situés à l'extérieur et sur la frontière du carré $Q = [(0,0), (0,1), (1,0), (1,1)]$ et soit Ψ l'ensemble de ceux-là parmi les points intérieurs à Q dont l'ordonnée n'appartient pas à l'ensemble parfait de Cantor (= ensemble des nombres réels situés sur $[0,1]$ et dont le développement triadique peut être fait sans utiliser le chiffre 1).

Désignons enfin par D un sous-ensemble (A) non-mesurable B de l'ensemble parfait de Cantor P , placé sur le segment $0 \leq y \leq 1$ de l'axe Oy .

D'après les résultats connus de SOUSLIN et de MM. LUSIN et SIERPINSKI l'ensemble D est une projection orthogonale d'un ensemble H du type G_δ , situé dans le carré Q . Par une contraction effectuée dans la seule direction de l'axe Ox on transforme H en un ensemble H_n de

¹⁾ Cette question a été posée par l'auteur en 1923 (dans sa communication à la Société Mathématique de Moscou).

points dont les ordonnées sont restées les mêmes, tandis que les abscisses ont leurs valeurs contenues entre $x = \frac{1}{2n}$ et $x = \frac{1}{2n+1}$.

L'ensemble

$$K = \Phi + \Psi + \sum_{n=1}^{\infty} H_n$$

est un ensemble G_δ : en effet, Φ étant fermé et Ψ ouvert, ces deux ensembles sont évidemment des G_δ ; quant à $\sum_{n=1}^{\infty} H_n$, cet ensemble est somme d'une infinité dénombrable d'ensembles G_δ situés dans des régions du plan deux à deux séparées, il est donc aussi un ensemble G_δ ; par conséquent K est somme de trois ensembles G_δ ; il entre donc encore dans la même classe d'ensembles.

On voit de suite que tout point accessible de K situé sur l'axe Oy appartient à l'ensemble parfait P , tout en étant étranger à l'ensemble D .

Or tout point de $P-D$ est évidemment accessible au sens étroit, de sorte qu'on a, pour l'ensemble K :

$$Y.L = Y.L_d = P - D$$

(en désignant par Y l'ensemble de tous les points de l'axe Oy).

L'ensemble $P-D$ n'étant pas un ensemble (A) (dans le cas contraire les deux ensembles $P-D$ et D seraient mesurables B), il en est de même pour les ensembles L et L_d , c. q. f. d.

Mathematics. — “Zur allgemeinen Dimensionstheorie”. By L. TUMARKIN.
(Communicated by Prof. L. E. J. BROUWER).

(Communicated at the meeting of November 28, 1925.)

1. Zweck der vorliegenden Bemerkungen ist festzustellen, dass *wesentliche*, bisher nur für *abgeschlossene* Teilmengen eines kompakten metrischen Raumes bewiesene *dimensionstheoretische Sätze* für *beliebige* Teilmengen eines kompakten metrischen Raumes, also nach einem bekannten URYSOHNSchen Satze ¹⁾, für jeden separablen ²⁾ metrisierbaren, d.h., einem in Erweiterung eines URYSOHNSchen Theorems von TYCHONOFF bewiesenen Metrisationssatze ³⁾ zufolge, *für einen beliebigen regulären* ⁴⁾, dem II. Abzählbarkeitsaxiome ⁵⁾ genügenden topologischen Raum gelten. Ausserdem gebe ich einen ganz neuen Satz, der die Dimensionstheorie beliebiger separabler metrisierbarer Räume auf die Untersuchung allein *vollständiger* metrischer Räume zurückführen lässt.

2. Den eigentlichen Kern meiner ganzen Untersuchung bildet ein Hilfsatz, der nichts anderes als eine direkte Verallgemeinerung des bekannten HEINE-BOREL-LEBESGUESchen Ueberdeckungssatzes ist, und zu dessen Formulierung ich eine Hilfsdefinition brauche.

Definition. — Es sei M irgend eine im kompakten metrischen Raume R liegende Menge, und \mathfrak{P} ein die Menge M als Vereinigungsmenge besitzendes System (beliebiger Mächtigkeit) von Mengen G , von denen jede ein Gebiet (rel. M) ⁶⁾ ist. Dann soll das Mengensystem \mathfrak{P} eine *kanonische Ueberdeckung* der Menge M heissen, sobald es folgender Bedingung genügt:

¹⁾ “Jeder separable ²⁾ metrisierbare topologische Raum ist einer Teilmenge eines (und desselben) kompakten metrischen Raumes homöomorph”; bewiesen in “Der Hilbertsche Raum u.s.w.”, Math. Ann. 92.

²⁾ Separabel heisst ein Raum falls er eine abzählbare dichte Teilmenge enthält (FRÉCHET).

³⁾ P. URYSOHN, “Zum Metrisationsproblem”, Math. Ann. 94; TYCHONOFF, Math. Ann. 95.

⁴⁾ Ein topologischer Raum heisst regulär (ALEXANDROFF—URYSOHN, “Zur Theorie der topologischen Räume”, Math. Ann. 92) falls jede Umgebung eines beliebigen Punktes die abgeschlossene Hülle einer gewissen Umgebung desselben Punktes enthält.

⁵⁾ HAUSDORFF, Grundzüge der Mengenlehre, Leipzig 1914, S. 263.

⁶⁾ HAUSDORFF, op. cit. SS. 215. u. 240. Ueber weitere Bezeichnungen und Terminologie siehe z. B. die Einleitung zum URYSOHNSchen “Mémoire sur les multiplicités Cantoriennes” (Fund. Math. VII, SS. 49–64).

falls ξ ein beliebiger Häufungspunkt der Menge M ist, so gibt es eine positive Zahl ε und ein zu \mathfrak{P} gehörendes G von der Beschaffenheit, dass

$$M \cdot S(\xi, \varepsilon) \subset G$$

ist.

Nun lautet unser Hilfssatz folgendermassen:

Ueberdeckungssatz. Falls eine kanonische Ueberdeckung \mathfrak{P} einer beliebigen Teilmenge M eines kompakten metrischen Raumes R vorliegt, so ist die Menge M bereits in endlich vielen Gebieten dieser Ueberdeckung enthalten.

Durch systematische Anwendung dieses Satzes lassen sich unter wiederholter Benutzung der in den Kapiteln IV—VI des URYSOHNSchen „Mémoire sur les multiplicités Cantoriennes I“ (Fund. Math. VII u. VIII) dargestellten Methoden der Reihe nach folgende Sätze beweisen:

I. Falls M eine im kompakten metrischen Raume R liegende Menge, und ξ irgend ein Punkt des Raumes R ist, so ist $\dim(M + \xi) = \dim M$.

II. Falls R ein beliebiger separabler metrisierbarer Raum von der endlichen Dimension $n > 0$ ist, so ist die Menge $K_n(R)$ aller Punkte, in denen R die Dimension n hat, in sich dicht⁷⁾.

Desgleichen lassen sich verschiedene sonstige Sätze der Theorie der Dimensionskerne auf den Fall beliebiger separabler metrisierbarer Räume übertragen. Die Menge $K_n(R)$ kann in unsrem Falle allerdings auch abzählbar sein⁸⁾.

Weitere Sätze (die sich immer durch Kombination der URYSOHNSchen Beweismethoden mit unsrem Ueberdeckungssatze und dem Satze I ergeben) sind folgende:

III. Die BROUWERSche Dimensionsdefinition ist mit der URYSOHN—MENGERSchen Definition für alle metrisierbaren separablen Räume äquivalent⁹⁾.

IV. Falls R ein beliebiger metrisierbarer separabler Raum ist, und daselbst ein höchstens abzählbares System von (rel. R) abgeschlossenen, höchstens n -dimensionalen Mengen gegeben ist, so ist die Vereinigungsmenge aller dieser Mengen höchstens n -dimensional¹⁰⁾.

⁷⁾ Dieser Satz wurde für kompakte Räume ausgesprochen von URYSOHN (C. R. 175, S. 442), und bewiesen von URYSOHN (Fund. Math. VIII) sowie von Menger (Math. Ann. 95).

⁸⁾ Dies zeigt sich an einer von SIERPINSKI konstruierten Menge (Fund. Math. II, S. 81); eine nähere Untersuchung dieses Falles findet sich bei Menger (Wiener Ber. 133, S. 442).

⁹⁾ Diese Äquivalenz wurde hinsichtlich kompakter sowie kondensierter Räume bewiesen von BROUWER (diese Proceedings 27, S. 635); für die ersteren Räume kann sie übrigens als Korollar eines URYSOHNSchen Lemmas („Mémoire sur les multiplicités Cantoriennes“, Ch. VI, § 5, lemme I, Fund. Math. VIII) betrachtet werden.

¹⁰⁾ Für kompakte Räume wurde dieser Satz ausgesprochen von URYSOHN (C. R. 175, S. 442), und bewiesen von URYSOHN (Fund. Math. VIII), sowie von Menger (Monatshefte f. Math. u. Phys. 34).

Aus IV ergibt sich (genau wie der entsprechende Satz im URYSOHNschen Mémoire, Kap. VI)

V. Jeder n -dimensionale metrisierbare separable Raum lässt sich als Vereinigungsmenge von $n+1$ zueinander fremden nulldimensionalen Mengen darstellen.¹¹⁾

Da die Umkehrung von V noch von URYSOHN selbst bewiesen war, so ist die Dimension (im Falle wo sie endlich ist) für beliebige metrisierbare separable Räume um 1 weniger als die kleinste Zahl der nulldimensionalen Mengen, die zum Aufbau des Raumes notwendig sind. Diese Zahl ändert sich übrigens nicht, wenn man eventuell auch zueinander nicht fremde nulldimensionale Mengen zulässt.

Auf den Satz V fussend, kann man mit Hilfe eines LAVRENTIEFFschen Satzes¹²⁾ folgendes beweisen:

VI. Jede (in einem separablen vollständigen Raume R liegende) Menge M ist in einer G_δ -Menge derselben Dimension wie M enthalten.

Aus einem ALEXANDROFFschen Satze¹³⁾ ergibt sich dann sofort

VII. Jeder separable metrisierbare Raum R ist einer Teilmenge eines, dieselbe Dimension wie R besitzenden, vollständigen metrischen Raumes \tilde{R} homöomorph.

Indem wir die dem Raume R homöomorphe Teilmenge von \tilde{R} durch R^* bezeichnen, kann es vorkommen (wie Beispiele zeigen) dass $\tilde{R}-R^*$ notwendig von positiver Dimension ist.

Es sei hier nur auf diese Tatsache hingewiesen, an die, wie man sofort einsieht, interessante Problemstellungen anknüpfen.

Eine ausführlichere Darstellung der soeben besprochenen Ergebnisse habe ich am 4. Oktober 1925 der Moskauer Mathematischen Gesellschaft vorgelegt; sie erscheint übrigens demnächst in den „Mathematischen Annalen“.

¹¹⁾ C. R. 175, S. 442, sowie Fund. Math. VIII.

¹²⁾ LAVRENTIEFF, Contribution à l'étude des ensembles homéomorphes. Fund. Math. VI (der Satz lässt sich unmittelbar auf in vollständigen Räumen gelegene Mengen übertragen).

¹³⁾ P. ALEXANDROFF, Sur les ensembles de 1^{re} classe et les espaces abstraits (C. R. 178).

Mathematics. — “Über stetige Abbildungen kompakter Räume”. By PAUL ALEXANDROFF. (Communicated by Prof. L. E. J. BROUWER).

(Communicated at the meeting of November 28, 1925).

1. Wir wollen im folgenden eine topologische Eigenschaft (d. h. Eigenschaft eines topologischen Raumes) eine *starke* Eigenschaft nennen, falls aus ihrer Geltung für irgend einen topologischen Raum R folgt, dass sie gleichzeitig in jedem topologischen Raume $R^* = f(R)$, der ein stetiges Bild des Raumes R ist, erfüllt ist ¹⁾.

So sind die Eigenschaften eines topologischen Raumes bzw. kompakt, bikompakt, zusammenhängend zu sein, starke topologische Eigenschaften, ebenso wie z. B. die Eigenschaft eine abzählbare dichte Teilmenge zu besitzen.

Dagegen gehört weder das 1. Abzählbarkeitsaxiom, noch, wie Herr TYCHONOFF durch scharfsinnige Beispiele gezeigt hat, das 2. Abzählbarkeitsaxiom, noch irgendeines der Trennungsaxiome (also insbesondere weder Regularität noch Normalität), noch Metrisierbarkeit, noch verschiedene Kombinationen dieser Verhältnisse zu den starken Eigenschaften. Auch braucht, sogar im Falle dass die beiden Räume, R und sein stetiges Bild R^* , kompakt sind, keineswegs die Bildmenge F^* einer in R abgeschlossenen Menge F in R^* abgeschlossen zu sein.

Das Verhalten der topologischen Räume zu stetigen Abbildungen darf also nicht als ein selbstverständliches betrachtet werden; es bedarf vielmehr einer sorgfältigen Untersuchung.

2. Ich will hier einige Ergebnisse dieser Untersuchung für den Fall der kompakten Räume mitteilen ²⁾. Es gelten zuerst die Sätze:

1. Falls R ein bikompakter ³⁾ topologischer Raum und R^* ein stetiges

¹⁾ Der Ausdruck “topologischer Raum” wird überall in dieser Arbeit in dem von HAUSDORFF (Grundzüge der Mengenlehre, Leipzig 1914) gebrauchten Sinne gemeint. Das Gleiche gilt von den übrigen topologischen Begriffen: stetige Abbildung, Abzählbarkeitsaxiome, Zusammenhang u.s.w. Vgl. über Bezeichnungen auch die Einleitung zum URYSOHNschen “Mémoire sur les multiplicités Cantoriennes”, I. Teil (Fund. Math. VII, SS. 49–64). Man könnte auch in einem festen Raume R gelegene Mengen betrachten, und dann eine Eigenschaft dieser Mengen stark nennen, falls sie für jede (in R gelegene) stetige Bildmenge einer Menge M erfüllt ist, sobald sie für die Menge M selbst zutrifft.

So ist z. B. für einen Euklidischen Raum die Eigenschaft einer Menge, eine beschränkte abgeschlossene oder auch eine (A) -Menge zu sein eine in bezug auf diesen Raum starke Eigenschaft.

²⁾ Eine ausführliche Darstellung der den Inhalt dieser Voranzeige bildenden Sätze erscheint demnächst in den „Mathematischen Annalen“.

³⁾ Vgl. P. ALEXANDROFF u. P. URYSOHN, Zur Theorie der topologischen Räume, Math. Ann. 92.

Bild von R ist, so ist auch R^* bikompakt und jeder abgeschlossenen Teilmenge von R entspricht eine abgeschlossene Teilmenge von R^* ⁴⁾.

II. Die Eigenschaft eines topologischen Raumes, kompakt und metrisierbar zu sein, ist eine starke Eigenschaft.

3. Die bikompakten Räume verhalten sich besonders einfach, und zwar lässt sich jeder topologische Raum R^* , der ein stetiges Bild des bikompakten Raumes R ist, direkt aus abgeschlossenen Teilmengen des Raumes R erbauen.

Das geschieht mit Hilfe folgender Erklärung:

DEFINITION. Die Zerlegung $R = \Sigma X$ des topologischen Raumes R in zueinander fremde abgeschlossene Mengen X heisst stetig, falls folgende Bedingung erfüllt ist: es sei X_0 eine beliebige der Mengen X , und G irgend ein die Menge X_0 enthaltendes Gebiet. Dann gibt es ein die Menge X_0 ebenfalls enthaltendes Gebiet G_0 von der Art, dass jede Menge X unserer Zerlegung in G enthalten ist, falls sie zu G_0 nicht fremd ist.

Jede stetige Zerlegung

$$(1) \quad R = \Sigma X$$

eines topologischen Raumes R induziert einen neuen Raum R^* (der im allgemeinen ein FRÉCHETScher H -Raum ist), und zwar erhält man den Raum R^* indem man alle Punkte jeder Menge X_0 zu einem "Punkte" x_0^* identifiziert, und als Umgebung $U(x_0^*)$ die Gesamtheit aller "Punkte" x^* betrachtet, für die die entsprechenden Mengen X in einem (beliebigen) die Menge X_0 enthaltenden Gebiete G (des Raumes R) gelegen sind ⁵⁾.

Es gilt nun folgender Satz:

III. Ein stetige Zerlegung eines bikompakten topologischen Raumes R induziert stets einen bikompakten topologischen Raum R^* . Der Raum R^* ist dabei ein stetiges Bild von R .

Falls umgekehrt R^* ein stetiges Bild des bikompakten topologischen Raumes R ist, so erhält man eine stetige Zerlegung

$$R = \Sigma X$$

von R , indem man durch X die Menge aller Punkte von R bezeichnet, die in einen und denselben Punkt x^* von R^* abgebildet werden.

4. Der Satz III gilt insbesondere für kompakte metrische Räume, und erlaubt den Begriff einer topologischen Eigenschaft stetiger Zerlegungen eines solchen Raumes festzustellen: so kann man z. B. eine stetige

⁴⁾ Den zweiten Teil dieses Satzes habe ich zuerst nur für dem 1. Abzählbarkeitsaxiom genügende Räume bewiesen; Herr VEDENISSOFF hat mich auf die allgemeine Gültigkeit dieses Satzes aufmerksam gemacht.

⁵⁾ So induziert z.B. die stetige Zerlegung einer kompakten Punktmenge R eines Euklidischen Raumes in ihre „Stücke“ als Raum R^* eine kompakte nirgends dichte Punktmenge des Linearkontinuums (vgl. BROUWER, diese Proceedings, April 1911).

Zerlegung (1) eines gegebenen Raumes R n -dimensional nennen, falls der durch diese Zerlegung induzierte Raum R^* n -dimensional ist. Insbesondere ist folgender Satz leicht zu beweisen:

IV. *Die Zerlegung eines beliebigen kompakten metrischen Raumes in die Menge seiner Komponenten ist eine stetige nulldimensionale Zerlegung.*

5. Es sei endlich folgender Fundamentalsatz erwähnt:

V. *Jeder kompakte metrische Raum ist ein stetiges Bild einer (beliebigen) perfekten nulldimensionalen Menge (also z. B. der CANTORSchen Dreiteilungsmenge).*

Die Sätze II, III und V liefern zusammen das folgende Ergebnis:

Einen kompakten metrischen Raum definieren, sagt genau so viel aus, wie eine stetige Zerlegung der CANTORSchen perfekten Menge angeben.

6. Ich möchte zum Schluss auf folgendes Problem hinweisen. Es scheint sehr plausibel zu sein, dass jeder kompakte metrische Raum, für den eine stetige nulldimensionale Zerlegung in höchstens n -dimensionale abgeschlossene Mengen vorhanden ist, auch selbst höchstens n -dimensional ist. Zuzufolge des Satzes IV würde die positive Lösung dieses Problems u. a. bedeuten, dass jeder n -dimensionale kompakte Raum ein n -dimensionales Kontinuum enthält, was eine URYSOHNSche Vermutung, die für die gesamte Dimensionstheorie von grosser Wichtigkeit zu sein scheint, beweisen würde.

Man könnte natürlich auch weitere Probleme an die obigen Entwicklungen anknüpfen.

Zusatz bei der Korrektur. Wie ich soeben ersehe, behandelt Herr R. L. MOORE in seiner Arbeit: "Concerning upper semicontinuous collections of continua" (Amer. Trans. 27 (1925), pp. 416—428) einen Begriff, der sich mit dem obigen Begriff der stetigen Zerlegung für den Fall metrischer Räume mit Kontinuen als Zerlegungseinheiten im wesentlichen deckt. Da aber Herr MOORE sich a. a. O. auf ebene Kontinua beschränkt, so kommen seine Resultate mit den meinigen nicht weiter in Berührung.

Mathematics. — “Ueber Zerlegungen kompakter metrischer Räume in zueinander fremde abgeschlossene Mengen.” By L. TUMARKIN.
(Communicated by Prof. L. E. J. BROUWER).

(Communicated at the meeting of November 28, 1925.)

Herr P. ALEXANDROFF hat den Begriff der stetigen und insbesondere der stetigen nulldimensionalen Zerlegung eines kompakten metrischen Raumes in zueinander fremde abgeschlossene Mengen¹⁾ eingeführt und ein sich dazuf beziehendes Dimensionsproblem²⁾ aufgestellt. Im folgenden will ich die positive Lösung dieses Problems mitteilen. Es handelt sich also um den Beweis des folgenden Satzes:

I. Falls eine stetige nulldimensionale Zerlegung des kompakten metrischen Raumes R in zueinander fremde, abgeschlossene, höchstens n -dimensionale Mengen X vorliegt, so ist der Raum R höchstens n -dimensional.

Der Beweis dieses Satzes stützt sich auf folgenden Hilfssatz:

Es sei L ein separabler metrischer nulldimensionaler Raum;

$$\Phi_1, \Phi_2, \dots, \Phi_s$$

ein System von zueinander fremden, in L abgeschlossenen Teilmengen des Raumes L ; endlich,

$$G_1^0, G_2^0, \dots, G_s^0$$

ein System von Gebieten (rel. L), die so beschaffen sind, dass

$$L = \sum_{m=1}^s G_m^0 \text{ und } G_m^0 \supset \Phi_m \quad (\text{für alle } m, 1 \leq m \leq s)$$

ist. Dann lassen sich s zueinander fremde, den Bedingungen

$$L = \sum_{m=1}^s G_m \text{ und } G_m^0 \supset G_m \supset \Phi_m$$

genügende Gebiete (rel. L) bestimmen.

Nachdem dieser Hilfssatz bewiesen ist, betrachten wir den durch die gegebene Zerlegung von R induzierten, aus Punkten $x^* \in X$ gebildeten kompakten nulldimensionalen Raum R^* .

Es sei nun ε eine beliebige positive Zahl, und X eine beliebige der

1) P. ALEXANDROFF, “Ueber stetige Abbildungen kompakter Räume”, im gleichen Band dieser Proceedings, S. 997. Es sei auf diese Arbeit auch wegen Terminologie und Bezeichnungen hingewiesen.

2) Siehe die unter 1) zitierte Arbeit, namentlich § 6.

Mengen, in die R zerlegt ist. Es existiert³⁾ eine $(\varepsilon, n+1)$ -Überdeckung⁴⁾

$$(1) \quad \Phi_X^1, \Phi_X^2, \dots, \Phi_X^{k_X}$$

jeder Menge X , und eine positive Zahl σ_X , die so klein ist, dass das Mengensystem

$$(2) \quad \bar{S}(\Phi_X^i, \sigma_X), \quad 1 \leq i \leq k_X$$

von der Ordnung $\leq n+1$ ist⁵⁾, und ausserdem $\delta\{\bar{S}(\Phi_X^i, \sigma_X)\} < \varepsilon$ bleibt.

Es sei X_0 eine beliebige der Mengen X . Wir bezeichnen durch $G_{x_0}^*$ das im Raume R^* gelegene Gebiet, das aus allen denjenigen x^* gebildet ist, denen in $S(X_0, \sigma_{X_0})$ enthaltene Mengen X entsprechen.

Der ganze Raum R^* wird dann durch endlichviele Gebiete G_{x^*} überdeckt. Es seien

$$G_{x_1^*}, G_{x_2^*}, \dots, G_{x_s^*},$$

diese Gebiete. Indem wir die $G_{x_i^*}$, $1 \leq i \leq s$, als G_i^0 und die Punkte x_i^* als Φ_i des Hilfssatzes betrachten, und diesen Hilfssatz anwenden, erhalten wir in R^* die Gebiete

$$G_1^*, G_2^*, \dots, G_s^*,$$

die zueinander fremd sind und den Bedingungen

$$x_i^* \in G_i^* \subset G_{x_i^*}^* \quad (1 \leq i \leq s)$$

genügen. Die G_i^* sind aber gleichzeitig in R^* abgeschlossen. Die ihnen entsprechenden Mengen $F_i \subset R$ sind also in R abgeschlossen und zueinander fremd. Ausserdem ist

$$X_i \subset F_i \subset S(X_i, \sigma_{X_i}).$$

Indem wir durch F_{ik} die Mengen

$$F_{ik} = F_i \cdot \bar{S}(\Phi_{X_i}^k, \sigma_{X_i}), \quad 1 \leq k \leq k_{X_i}, \quad 1 \leq i \leq s$$

bezeichnen, erhalten wir, wie leicht ersichtlich, eine $(\varepsilon, n+1)$ -Überdeckung des Raumes R (mittels der Mengen F_{ik}). Da ε beliebig klein genommen werden konnte, ergibt sich unser Satz I direkt aus einem URYSOHNschen Satze.⁵⁾

Aus dem Satze I und einem Resultate Herrn ALEXANDROFF's⁶⁾ ergibt sich nun sofort folgende Lösung eines vorhin von URYSOHN gestellten Problems:

II. *Jeder endlichdimensionale kompakte metrische Raum enthält ein Kontinuum von der gleichen Dimension.*

³⁾ Siehe hierzu das Kapitel V des URYSOHNschen "Mémoire sur les multiplicités Cantorienes" (Fund. Math. VIII), sowie den einschlägigen Aufsatz von Menger (Monatshefte für Math. u. Phys. 34).

⁴⁾ Eine $(\varepsilon, n+1)$ -Überdeckung der Menge X ist ein Mengensystem (1), wobei die Φ_X^i ($1 \leq i \leq k_X$) abgeschlossen, und so beschaffen sind, dass

$$\sum_{i=1}^{k_X} \Phi_X^i = X, \quad \delta(\Phi_X^i) < \varepsilon,$$

ist, und es keinen Punkt gibt, der mehr als $n+1$ unter den Mengen (1) angehört.

⁵⁾ Vgl. den unter ³⁾ zitierten URYSOHNschen Mémoire.

⁶⁾ Die unter ¹⁾ zitierte Arbeit, Satz IV.

[illegible]

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